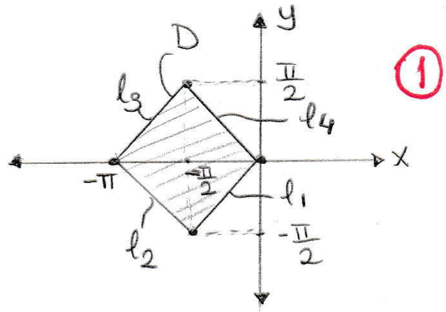
Soru 1: D bölgesi köşeleri $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $(-\pi,0)$, $(0,0)$, $\left(-\frac{\pi}{2},-\frac{\pi}{2}\right)$ noktalarında bulunan kare olmak üzere,

$\iint_D (x+y)^2 \cdot \sin(y-x)\,dydx$ integralini hesaplayınız. (20P)



Çözüm:

$l_1 : (0,0) \to \left(-\frac{\pi}{2},-\frac{\pi}{2}\right)$

$$\frac{y-0}{0-(-\frac{\pi}{2})} = \frac{x-0}{0-(-\frac{\pi}{2})} \Rightarrow y = x \quad 1$$

$l_2 : (-\pi,0) \to \left(-\frac{\pi}{2},-\frac{\pi}{2}\right)$

$$\frac{y-0}{0-(-\frac{\pi}{2})} = \frac{x-(-\pi)}{-\pi-(-\frac{\pi}{2})} \Rightarrow y = -x-\pi \quad 1$$

$l_3 : (-\pi,0) \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

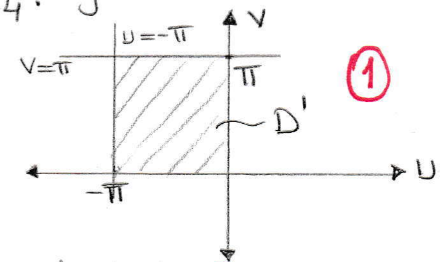
$$\frac{y-0}{0-\frac{\pi}{2}} = \frac{x-(-\pi)}{-\pi-(-\frac{\pi}{2})} \Rightarrow y = x+\pi \quad 1$$

$l_4 : (0,0) \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\frac{y-0}{0-\frac{\pi}{2}} = \frac{x-0}{0-(-\frac{\pi}{2})} \Rightarrow y = -x \quad 1$$

$$\left.\begin{aligned} x+y &= u \\ y-x &= v \end{aligned}\right] \quad 1$$

D		D'	
$l_1: y-x=0$	$\to$	$v=0$	1
$l_2: y+x=-\pi$	$\to$	$u=-\pi$	1
$l_3: y-x=\pi$	$\to$	$v=\pi$	1
$l_4: y+x=0$	$\to$	$u=0$	1



$$J(u,v) = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{1}{2} \quad 1 \qquad dxdy = \frac{1}{2}\,dudv \quad 1$$

$$I = \iint_{D'} u^2 \cdot \sin v \cdot \frac{1}{2}\,dudv = \int_{u=-\pi}^{u=0}\int_{v=0}^{v=\pi} \frac{1}{2}\sin v \cdot u^2\,dvdu \quad 1\ 1\ 1\ 1\ 1$$

$$= \int_{-\pi}^{0} (-\cos v)\Big|_0^{\pi} \frac{1}{2} u^2\,du = \int_{-\pi}^{0} (1\cancel{+1}) \cdot \frac{1}{\cancel{2}}\, u^2\,du \quad 1$$

$$= \frac{u^3}{3}\Big|_{-\pi}^{0} = \frac{\pi^3}{3} \quad 1$$

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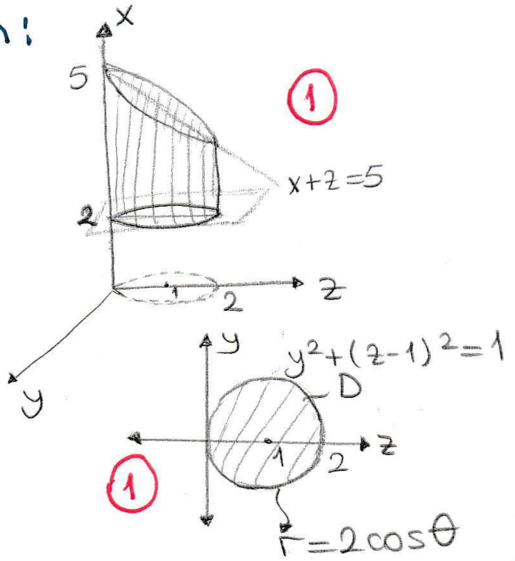
Soru 2: $y^2+(z-1)^2=1$, $x=2$ ve $x+z=5$ yüzeyleri ile oluşturulmuş kapalı cismin hacmini iki katlı kutupsal integral ile hesaplayın. **(20P)**

Çözüm:



$$V=\iint_D (x_1-x_2)\,dy\,dz$$

$$V=\iint_D (5-z-2)\,dy\,dz$$

$$V=\iint_D (3-z)\,dy\,dz \quad 2$$

$z=r\cos\theta$
$y=r\sin\theta$ 2
$dydz=r\,dr\,d\theta$

$$V=\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{2\cos\theta}(3-r\cos\theta)\,r\,dr\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}(3r-r^2\cos\theta)\,dr\,d\theta$$

1 1 2 1 1

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{3}{2}r^2-\frac{r^3}{3}\cos\theta\right)\Big|_0^{2\cos\theta}d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(6\cos^2\theta-\frac{8}{3}\cos^4\theta\right)d\theta \quad 2$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[6\left(\frac{1+\cos2\theta}{2}\right)-\frac{8}{3}\left(\frac{1+\cos2\theta}{2}\right)^2\right]d\theta \quad 2$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(3+3\cos2\theta-\frac{2}{3}-\frac{4}{3}\cos2\theta-\frac{2}{3}\left(\frac{1+\cos4\theta}{2}\right)\right)d\theta$$

$$=2.\left(2\theta+\frac{5}{6}\sin2\theta-\frac{1}{12}\sin4\theta\right)\Big|_0^{\frac{\pi}{2}} \quad 2$$

$$=2.2.\frac{\pi}{2}=2\pi\ br^3 \quad 2$$

Soru 3: $\{a_n\}_{n=1}^{\infty} = \left\{ \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{9}{32}, \cdots \right\}$ dizisi verilsin.

Buna göre $\sum_{n=1}^{\infty} a_n$ serisinin toplamını hesaplayınız. (15 P)

Çözüm: $\{a_n\} = \left\{ \frac{2n+1}{2^{n+1}} \right\}$ 1

$$\sum_{n=1}^{\infty} \frac{2n+1}{2^{n+1}} = \underbrace{\sum_{n=1}^{\infty} \frac{n}{2^n}}_{A} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^{n+1}}}_{B} \quad 2$$

$$B = \sum_{n=1}^{\infty} \frac{1}{4} \cdot \left(\frac{1}{2}\right)^{n-1} \Rightarrow \begin{cases} a = \frac{1}{4} \\ r = \frac{1}{2} < 1 \end{cases} \quad \text{Geometrik seri} \quad \frac{a}{1-r}$$

$$= \frac{\frac{1}{4}}{1-\frac{1}{2}} \; 2 \; = \frac{1}{2}, \quad 1$$

$$A = \sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^n \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (|x|<1) \quad 2$$

$\downarrow$ Türev alalım

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{1}{(1-x)^2} \quad (|x|<1) \quad 2$$

$\downarrow$ x ile çarpalım

$$\sum_{n=1}^{\infty} n \cdot x^n = \frac{x}{(1-x)^2} \quad (|x|<1) \quad 2$$

$$A = \sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} \; 1 \qquad \left(\frac{1}{2} < 1\right)$$

$$= 2, \quad 1$$

$$\sum_{n=1}^{\infty} \frac{2n+1}{2^{n+1}} = A + B = 2 + \frac{1}{2} = \frac{5}{2} \quad 1 \quad //$$

Soru 4: $\sum_{n=1}^{\infty}\left(\sqrt[n]{2020}-1\right)$ serisinin karakterini belirleyiniz. (10P)

Çözüm:

$\sum_{n=1}^{\infty}\frac{1}{n}$ harmonik serisini (ıraksak) seçelim. 1

Limit karşılaştırma testini uygularsak,

$$\lim_{n\to\infty}\frac{(2020)^{1/n}-1}{\frac{1}{n}}=\frac{0}{0} \quad 1$$

$$\overset{L'H}{=}\lim_{n\to\infty}\frac{\cancel{-\frac{1}{n^2}}\cdot\underbrace{(2020)^{1/n}}_{1}\cdot\ln(2020)}{\cancel{-\frac{1}{n^2}}}=\ln(2020)\neq 0,\infty$$

3 1 1

olduğundan iki seri aynı karakterdedir. 1

$\sum_{n=1}^{\infty}\frac{1}{n}$ ıraksak olduğundan,

$\sum_{n=1}^{\infty}\left(\sqrt[n]{2020}-1\right)$ serisi ıraksaktır. 2

Soru 5: $P(1,1,1)$ bir nokta, $f(x,y,z)$ ve $g(x,y,z)$ aşağıdaki koşulları sağlayan, türevlenebilen iki fonksiyon olsun.

I) $f(P)=2$ ve $g(P)=5$

II) $\left.\dfrac{\partial g}{\partial x}\right|_P=-1$

III) f nin P noktasındaki en hızlı artışının gerçekleştiği yön $\vec{u}=\vec{i}-8\vec{j}+4\vec{k}$ ve bu yöndeki türevinin değeri 6 dır.

IV) $f(x,y,z)+2g(x,y,z)=12$ yüzeyinin P noktasındaki teğet düzleminin denklemi $4x+2y+3z=9$ dur.

Buna göre $\left.\dfrac{\partial g}{\partial y}\right|_P$ değerini hesaplayınız. **(20 P)**

Çözüm: $\vec{u}=\langle 1,-8,4\rangle \qquad \nabla f|_P \,//\, \vec{u}$

$$\nabla f|_P=\langle k,-8k,4k\rangle \quad 2$$

$$|\nabla f|_P|=\sqrt{k^2+64k^2+16k^2}=6 \quad 1 \quad \Rightarrow 81k^2=36 \Rightarrow k=\frac{6}{9}=\frac{2}{3} \quad 1$$

$$\nabla f|_P=\left\langle \frac{2}{3},-\frac{16}{3},\frac{8}{3}\right\rangle=\langle f_x|_P, f_y|_P, f_z|_P\rangle \quad 3$$

$$H:\ f(x,y,z)+2g(x,y,z)-12=0$$

$$\nabla H|_P=\langle (f_x+2g_x)|_P, (f_y+2g_y)|_P, (f_z+2g_z)|_P\rangle \;3\; =\langle 4c,2c,3c\rangle \quad 2$$

$$f_x|_P+2g_x|_P=4c$$

$$\frac{2}{3}+2\cdot(-1)=4c \quad 2 \quad \Rightarrow -\frac{4}{3}=4c \Rightarrow c=-\frac{1}{3} \quad 2$$

$$f_y|_P+2g_y|_P=2c$$

$$-\frac{16}{3}+2g_y|_P=2\cdot\left(-\frac{1}{3}\right) \quad 2$$

$$2g_y|_P=-\frac{2}{3}+\frac{16}{3}=\frac{14}{3} \Rightarrow \left.\frac{\partial g}{\partial y}\right|_P=\frac{7}{3} \quad 2$$

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Soru 6: $f(1,3)=-2$, $f(1.02, 2.99)=-1.98$ şartlarını sağlayan bir $f(x,y)$ fonksiyonunu göz önüne alalım. Bu fonksiyonun $P(1,3)$ noktasındaki $\vec{u}=-4\vec{i}+2\vec{j}$ yönündeki türevinin değerini yaklaşık olarak hesaplayınız. **(15P)**

Çözüm:

$$f(x,y)\approx L(x,y)=f(1,3)+f_x(1,3).(x-1)+f_y(1,3).(y-3) \quad 2$$

$$f(1.02, 2.99)\approx -2+f_x(1,3).(1.02-1)+f_y(1,3)(2.99-3)$$

$$-1.98\approx -2+f_x|_p\cdot(0.02)+f_y|_p\cdot(-0.01) \quad 2$$

$$0.02\approx 0.01(2f_x|_p-f_y|_p)$$

$$\Rightarrow \quad 2f_x|_p-f_y|_p\approx 2 \quad 2$$

$$\frac{\vec{u}}{|\vec{u}|}=\frac{\langle -4,2\rangle}{2\sqrt{5}}=\langle -\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\rangle \quad 2$$

$$D_{\vec{u}}f|_p=\frac{\vec{u}}{|\vec{u}|}\cdot\nabla f|_p \overset{1}{=} \langle -\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\rangle\cdot\langle f_x|_p, f_y|_p\rangle \quad 2$$

$$=\frac{-1}{\sqrt{5}}(2f_x|_p-f_y|_p)^{2}\approx\frac{-2}{\sqrt{5}} \quad 2$$

(*)

Not:

$\vec{u}$ vektörünü birim yapmadan işleme devam edene (*) kısmından puan verilmeyecektir. Öğrenciye sorudan toplam <u>6 puan</u> verilecektir.