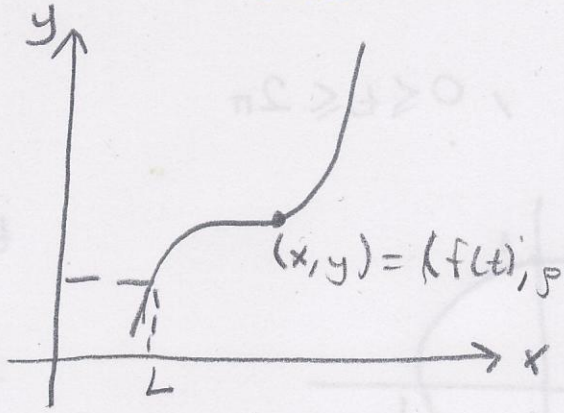
Parametrik Denklemler

$y = f(x)$

$x = f(t)$, $t \in I$
(c); $y = \rho(t)$



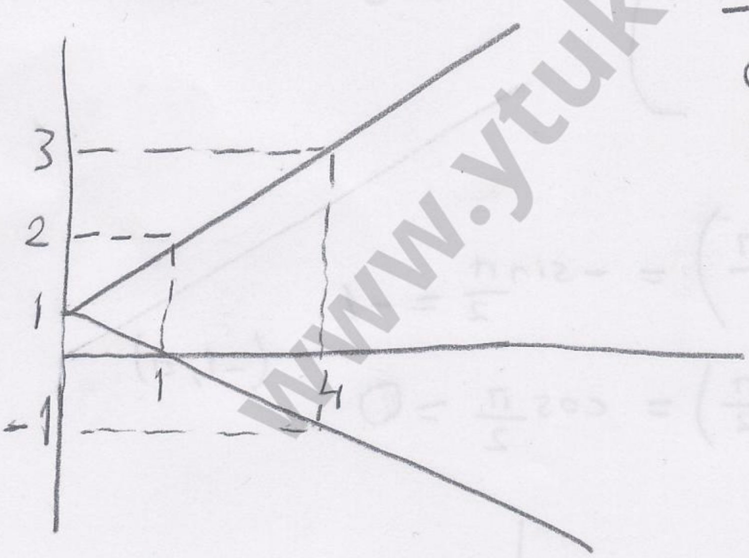
$a \leq t \leq b$

$(f(a), \rho(a)) \rightarrow$ Eğrinin başlangıç noktası

$(f(b), \rho(b)) \rightarrow$ Eğrinin bitiş noktası

örnek = $x = t^2$
$y = t+1$, $-\infty < t < \infty$
$\downarrow$
$t = y-1$
$x = (y-1)^2$

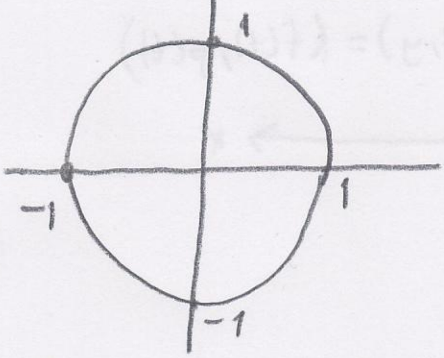
t	$x = t^2$	$y = t+1$
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2



Örnek =

$x = \cos t$

$y = \sin t \quad , \; 0 \leq t \leq 2\pi$



$t = 0 \Rightarrow \begin{matrix} x = 1 \\ y = 0 \end{matrix}$

$t = \frac{\pi}{2} \Rightarrow \begin{matrix} x = 0 \\ y = 1 \end{matrix}$

$t = \pi \Rightarrow \begin{matrix} x = -1 \\ y = 0 \end{matrix}$

$$\begin{array}{l} x^2 = \cos^2 t \\ + \; y^2 = \sin^2 t \\ \hline x^2 + y^2 = 1 \end{array}$$

Örnek =

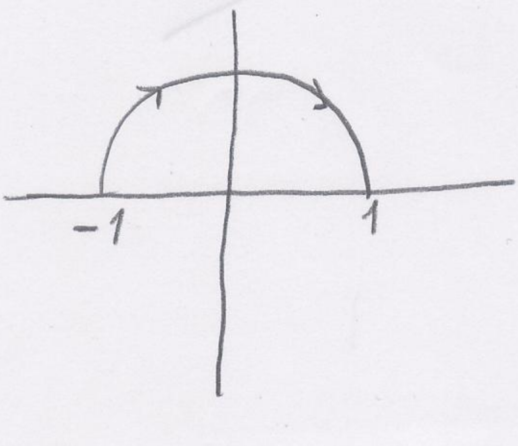
$$\left.\begin{array}{ll} \text{a) } x(t) = \sin t & \\ \quad\; y(t) = \cos t & , \; \frac{-\pi}{2} \leq t \leq \frac{\pi}{2} \\ & \\ \text{b) } x(t) = t - 1 & \\ \quad\; y(t) = \sqrt{2t - t^2} & 0 \leq t \leq 2 \end{array}\right\} \text{Aynı eğriyi temsil ederler.}$$

a) $t = \frac{-\pi}{2} \Rightarrow x = \sin\left(\frac{-\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

$y = \cos\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0 \quad (-1, 0)$

$t = 0 \Rightarrow \begin{matrix} x = 0 \\ y = 1 \end{matrix} \quad (0, 1)$

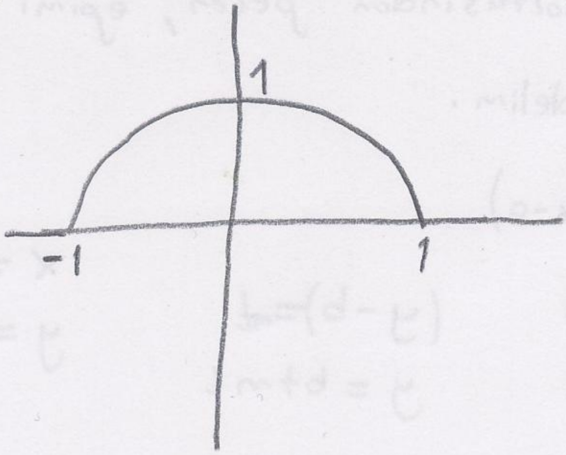
$t = \frac{\pi}{2} \Rightarrow \begin{matrix} x = 1 \\ y = 0 \end{matrix} \quad (1, 0)$



b) $t=0 \Rightarrow x=-1$
$\quad y=0$

$t=1 \Rightarrow x=0$
$\quad y=1$

$t=2 \Rightarrow x=1$
$\quad y=0$

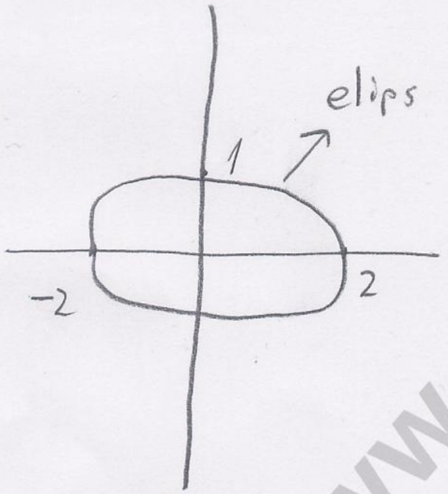


örnek

$x^2+2y+4z=4$ düzlemi
$x^2+4y^2=4$ silindiri

Arakesit eğrisini parametrize ediniz.

$$\frac{x^2}{4}+\frac{y^2}{1}=1$$



$x^2+4y^2=4 \Rightarrow$ parametrik edelim

$$\boxed{\begin{aligned} x&=2\cos t \\ y&=\sin t \end{aligned}}, \quad 0 < t \leq 2\pi$$

$$x+2y+4z=4$$

$$4z=4-x-2y$$

$$z=\frac{1}{4}(4-x-2y)$$

$$=1-\frac{1}{4}(x+2y)$$

$$=1-\frac{1}{4}(2\cos t+2\sin t)$$

$$=1-\frac{1}{2}(\cos t+\sin t)$$

$$z=\frac{1}{2}(\cos t+\sin t)$$

$x=2\cos t$
$y=\sin t$
$z=\frac{1}{2}(\cos t+\sin t)$
$, 0 \leq t \leq 2\pi$

örnek

(a,b) noktasından geçen, eğimi m olan doğruyu paramize edelim.

$$y-b=m(x-a)$$

$(x-a)=t$ $\quad$ $(y-b)=mt$ $\quad$ $x=a+t$

$x=a+t$ $\quad$ $y=b+mt$ $\quad$ $y=b+mt$, $-\infty<t<\infty$

örnek

$$x=t+\frac{1}{t}$$
$$y=t-\frac{1}{t} \quad , t>0$$

$x+y=2t$

$x-y=\frac{2}{t}$ $\Rightarrow x^2-y^2=4$

Teğetler ve Alanlar

$x=f(t)$

$y=g(t)$, $t\in I$

$\frac{dy}{dt}, \frac{dx}{dt}, \frac{dy}{dx}$ $\quad$ $y=h(x)$

$y=h[f(t)]=f(t)$

$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$ zincir kuralı

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\dot{y}}{\dot{x}} \Rightarrow \frac{d^2y}{dx^2}=\frac{dy^2}{dx}=\frac{dy^2}{dx}=\frac{dy^2}{dt}\cdot\frac{dt}{dx}$$

$$=\frac{\frac{dy^2}{dt}}{\frac{dx}{dt}} \quad \frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\cdot\left(\frac{\dot{y}}{\dot{x}}\right)}{\dot{x}}=\frac{\frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{(\dot{x})^2}}{\dot{x}}$$

$$=\frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{(\dot{x})^3}$$

örnek $(\sqrt{2}, 1)$ noktasındaki eğimini bulunuz. Teğet ve normal doğrunun denklemini bulunuz.

$x = \sec t$
$y = \tan t$, $\frac{-\pi}{2} < t < \frac{\pi}{2}$

$\sqrt{2} = \sec t = \frac{1}{\cos t}$
$\hookrightarrow t = \frac{\pi}{4}$

$1 = \tan t =$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{\sec t . \tan t} = \frac{\sec t}{\tan t}$$

$$m_T = y' \Big|_{t=\frac{\pi}{4}} = \frac{\sec\frac{\pi}{4}}{\tan\frac{\pi}{4}} = \sqrt{2}$$

Teğet denklemi $= y - 1 = \sqrt{2}.(x - \sqrt{2}) \Rightarrow y = \sqrt{2}x - 1$

$m_N . m_T = -1 \Rightarrow m_N = \frac{-1}{\sqrt{2}}$

Normal denklemi $\Rightarrow y - 1 = \frac{-1}{\sqrt{2}}(x - \sqrt{2})$

örnek

$$x = t - t^2$$
$$y = t - t^3 \quad \text{ise} \quad \frac{d^2y}{dx^2} = ?$$

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{1-3t^2}{1-2t}$$

$$\frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt} \cdot \left(\frac{1-3t^2}{1-2t}\right)}{1-2t}$$

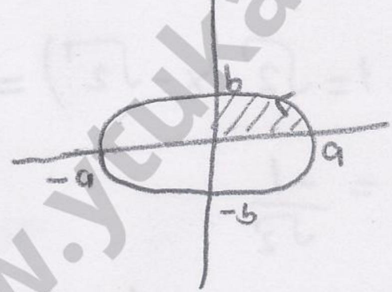
$$= \frac{\frac{-6t(1-2t) + 2(1-3t^2)}{(1-2t)^2}}{(1-2t)} = y'' \Rightarrow \underline{\frac{6t^2 - 6t + 2}{(1-2t)^3}}$$

örnek

$$x = a.\cos t$$
$$y = b.\sin t \quad , \quad 0 \leq t \leq 2\pi \quad \text{Elipsin alanını bulunuz.}$$

$a > b$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$A = 4 \cdot \int_0^a y.dx = 4 \int_{\pi/2}^{0} b.\sin t.(-a).\sin t.dt = 4ab \int_0^{\frac{\pi}{2}} \sin^2 t dt$$

$$= 4ab \int_0^{\frac{\pi}{2}} \frac{1-\cos 2t}{2} dt$$

$dx = -a.\sin t.dt$

$(a,b) \to (a,0) \longrightarrow \left.\begin{array}{l} a = a.\cos t \\ 0 = b.\sin t \end{array}\right\} \begin{array}{l} \cos t = 1 \\ \sin t = 0 \end{array} \quad t = 0$

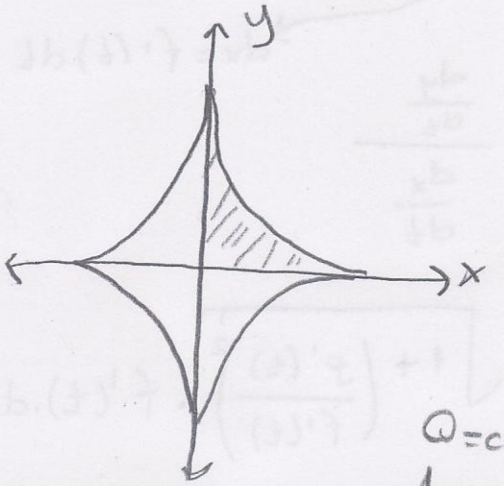
$\left.\begin{array}{l} 0 = a.\cos t \\ b = b.\sin t \end{array}\right\} \begin{array}{l} \cos t = 0 \\ \sin t = 1 \end{array}, \quad t = \frac{\pi}{2}$

$$= 2ab\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\pi/2}$$

$$A = 2ab\left(\frac{\pi}{2}\right) = a.b\pi.br^2$$

örnek

$x = \cos^3 t$
$y = \sin^3 t$, $0 \leq t \leq 2\pi$ parametrik denklemleri ile verilen astroid eğrisi ile sınırlı alanı bulunuz.



$$A = 4 \int_0^1 y\, dx$$

$dx = -3\cos^2 t \cdot \sin t \cdot dt$

$(0,1) \rightarrow (1,0)$

$0 = \cos^3 t$
$1 = \sin^3 t$
$t = \frac{\pi}{2}$

$1 = \cos^3 t$
$0 = \sin^3 t$
$t = 0$

$$= 4 \int_0^{\frac{\pi}{2}} \sin^3 t (-3) \cdot \cos^2 t \cdot \sin t \cdot dt = 12 \int_0^{\frac{\pi}{2}} \sin^4 t \cdot \cos^2 t \cdot dt$$

$$= \frac{3}{2} \int_0^{\frac{\pi}{2}} \left(1 - 2\cos 2t + \cos^2 2t\right) \cdot \left(1 + \cos 2t\right) dt$$

$\cos^2 2t = \frac{1 + \cos 4t}{2}$

$A = \frac{3\pi}{8}\ br^2$

PARAMETRİK OLARAK TANIMLI EĞRİNİN UZUNLUĞU

$y=f(x)\ ,\ a\leqslant x\leqslant b$

$$S=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Eğer C eğrisi $x=f(t)$, $y=\rho(t)$ $a\leqslant t\leqslant b$ ise parametrik olarak tanımlıysa $t=a$'dan $t=b$'ye artarken C eğrisi üzerinden sadece bir kez geçiyorsa C'nin uzunluğu

$$\begin{aligned}x&=f(t)\\ y&=\rho(t)\end{aligned}\ ,\ c\leqslant t\leqslant d \longrightarrow dx=f'(t).dt$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$S=\int_c^d\sqrt{1+\left(\frac{\rho'(t)}{f'(t)}\right)^2}\cdot f'(t).dt$$

$$S=\int_c^d\frac{\sqrt{[f'(t)]^2+[\rho'(t)]^2}}{\cancel{f'(t)}}\cdot \cancel{f'(t)}.dt$$

$$\longrightarrow L=S=\int_c^d\sqrt{\underbrace{(f'(t))^2}_{\frac{dx}{dt}}+\underbrace{(\rho'(t))^2}_{\frac{dy}{dt}}}$$

örnek

$$\begin{aligned}x&=r.\cos t\\ y&=r.\sin t\end{aligned}\ ,\ 0\leqslant t\leqslant 2\pi \quad \text{çemberinin uzunluğu}=?$$

$\frac{dx}{dt}=-r.\sin t$

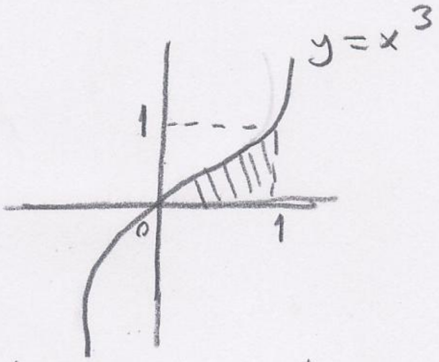
$\frac{dy}{dt}=r.\cos t$

$$S=\int_0^{2\pi}\sqrt{r^2\sin^2t+r^2\cos^2t}\,dt$$

$$=r\int_0^{2\pi}dt=rt\Big|_0^{2\pi}=\underline{2\pi r}$$

örnek $[0,1]$ aralığında $y=x^3$ eğrisinin altında kalan bölgenin alanını bulunuz.

$$x=t^2, \quad y=t^6, \quad -\infty < t < \infty$$



$$(0,0) \longrightarrow (1,1)$$

$$0=t^2 \qquad 1=t^2$$
$$0=t^6 \qquad 1=t^6$$
$$\underline{t=0} \qquad \underline{t=0}$$

$$A=\int_0^1 y\,dx = \int_0^1 t^6 . 2t . dt$$

$$=2\int_0^1 t^7 . dt$$

$$=2 . \frac{t^8}{8}\Big|_0^1 = \frac{2}{8} = \frac{1}{4}$$

örnek $x=\cos^3 t$, $y=\sin^3 t$, $0 \le t \le 2\pi$ astroid eğrisinin uzunluğunu bulunuz.

$$\frac{dx}{dt} = -3\cos^2 t . \sin t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 9\cos^4 t . \sin^2 t$$

$$\frac{dy}{dt} = 3\sin^2 t . \cos t \Rightarrow \left(\frac{dy}{dt}\right)^2 = 9\sin^4 t . \cos^2 t$$

$$+$$

$$= 9\cos^2 t . \sin^2 t$$
$$= 9\cos^2 t . \sin^2 t$$

$$\boxed{\begin{aligned} \sin 2t &= 2\sin t . \cos t \\ \sin t . \cos t &= \frac{\sin 2t}{2} \end{aligned}}$$

$$S = 4\int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t}\, dt = 12\int_0^{\pi/2} \sin t . \cos t\, dt$$

$$S = 12\int_0^{\pi/2} \frac{\sin 2t}{2} dt = 6\left(\frac{-\cos 2t}{2}\right)\Big|_0^{\pi/2} = -3(\cos\pi - 1)$$

$$= \underline{\underline{+6}}$$