$$x_1^{k+1} = (1-w)x_1^k + w(-1 + x_2^k - x_3^k)/3$$
$$x_2^{k+1} = (1-w)x_2^k + w(7 + x_1^{k+1} + x_3^k)/3$$
$$x_3^{k+1} = (1-w)x_3^k + w(-7 - x_1^{k+1} + x_2^{k+1})/3$$

Step 3: for $k = 0, 1, 2, \dots$ compute x^{k+1} from these equations, starting by first one. Computation for

$$x_1^1 = (1-w)x_1^0 + w(-1 + x_2^0 - x_3^0)/3 = (1-1{,}25).0 + \frac{1{,}25(-1+0-0)}{3}$$
$$= -0{,}41667$$

$$x_2^1 = (1-w)x_2^0 + \frac{w(7 + x_1^1 + x_3^0)}{3} = -0{,}25 \times 0 + \frac{1{,}25(7 - 0{,}41667 + 0)}{3}$$
$$= 2{,}7431$$

$$x_3^1 = (1-w)x_3^0 + w(-7 - x_1^1 + x_2^1) = -0{,}25 + \frac{1{,}25(-7 + 0{,}41667 + 2{,}7431}{3}$$
$$= -1{,}6001$$

The next three iterations are;

	x_1	x_2	x_3
x^2	1,4972	2,1880	−2,2288
x^3	1,0494	1,8782	−2,0141
x^4	0,9628	2,007	−1,9723
x^5	1	2	−2 (exact solution

Exam Q: $2x_1 - 3x_2 + 2x_3 = -11$

$x_1 + x_2 - 2x_3 = 8$

$3x_1 - 2x_2 - x_3 = -1$

find the solution by using Locobi / Gauss-Seider / SOR method, (use pivoting)

Clue = $\begin{matrix} 2 & & \\ & 1 & \\ & & -1 \end{matrix}$ $|2|+|1|+|-1| = 4$

$3x_1 - 2x_2 - x_3 = -1$

$2x_1 - 3x_2 + 2x_3 = -11$

$x_1 + x_2 - 2x_3 = 8$

$|3| + |-3| + |-2| = 8$

Exam Q: $x - 2y + 8z = 5$

$15x + 3y - 2z = 85$

$2x + 10y + z = 51$

find the solution by using pivoting with ... method.

Solving of Nonlineer Systems of Equations

$$x^2 - 3y = 5 \qquad e^x + \cos 2y = 3$$
$$xy + 2x - y = 0 \qquad x^2 + 3x(y+1) = 2$$

There are some methods to solve these nonlinear systems:

1) Fixed-point Iteration Method
2) Newton - Raphson Method

1) Fixed-Point Method

$$\left.\begin{aligned} x_{i+1} &= g_1(x_i, y_i, \ldots, z_i) \\ y_{i+1} &= g_2(x_i, y_i, \ldots, z_i) \\ z_{i+1} &= g_3(x_i, y_i, \ldots, z_i) \end{aligned}\right\}$$

Equation system has three variables. (x, y, z)

initial approximation (x_0, y_0, z_0)

Converpence Condition:

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| + \left|\frac{\partial g_1}{\partial z}\right| < 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| + \left|\frac{\partial g_2}{\partial z}\right| < 1$$

$$\left|\frac{\partial g_3}{\partial x}\right| + \left|\frac{\partial g_3}{\partial y}\right| + \left|\frac{\partial g_3}{\partial z}\right| < 1$$

Ex

Exam

$x_1^2 + x_1 . x_2 = 10$

$x_2 + 3x_1 . x_2^2 = 57$

find the solution by using f.PM $(x_1^0 = 1,5,\ x_2^0 = 3.5)$

$g_1(x)_{(x_1)} = \dfrac{10 - x_1^2}{x_2}$, $g_2(x)_{(x_2)} = 57 - 3x_1 . x_2^2$

$x_{1,0} = 1,5$, $x_{2,0} = 3,5$

#1 $x_{1,1} = \dfrac{10 - (1,5)^2}{3,5} = 2,21$ | $x_{2,1} = 57 - 3(1,5)(3,5)^2 = 1,87$

#2 $x_{1,2} = \dfrac{10 - (2,21)^2}{1,87} = 3,38$ | $x_{2,2} = 57 - 3(2,21)(1,87)^2 = 33,81$

#3 $x_{1,3} = \dfrac{10 - (3,38)^2}{33,81} = 1,3552$ | $x_{2,3} = 57 - 3(3,38)(33,81)^2 = -2,377.10^{-4}$

no convergence, no roots

$g_1(x)_{(x_1)} = \sqrt{10 - x_1 . x_2}$, $g_2(x)_{(x_2)} = \sqrt{\dfrac{57 - x_2}{3x_1}}$

Ex

$$\left.\begin{aligned} f_1(x,y) &= x^2-2x-y+0{,}5 \\ f_2(x_1,y) &= x^2+4y^2-4 \end{aligned}\right\} (x_0,y_0)=(0,1)$$

Sol

$$\rho_1(x) = x = \frac{x^2-y+0{,}5}{2}$$

$$\rho_2(x) = y = \frac{-x^2-4y^2+4+8y}{8} \qquad (x^2+4y^2-4+8y-8y)$$

$$(x^2+4y^2-4-8y=-8y$$

↓ draw

k	x	y
0	0	1
1	−0,25	1
2	−2,2187	0,9921
⋮		

Ex

$$\left.\begin{aligned} f_1(x,y) &= 1+y^2-4x^2 \\ f_2(x_1,y_0) &= 3+2x-x^2-y^2 \end{aligned}\right\} (x_0,y_0)=1,2$$

Sol

$$\rho_1(x) = x_{i+1} = \frac{8x_i+1+y^2-4x^2}{8} \qquad 1+y^2-4x^2+8x-8x$$

↘ draw x

$$\rho_2(x) = y_{i+1} = \frac{3+2x-x^2-y^2+2y}{2} \qquad 3+2x-x^2-y^2+2y-2y=0$$

↓ draw y

2- Newton - Raphson Methods

$$\left.\begin{array}{l} f_1(x, y, z, \ldots) = 0 \\ f_2(x, y, z, \ldots) = 0 \\ \vdots \\ f_n(x, y, z \ldots) = 0 \end{array}\right\} \begin{array}{l} f_1(x, y) = 0 \\ f_2(x, y) = 0 \end{array}$$

Outline of method for only two nonlineer equations

Step 1. Evoluate the functions

$$f(P_k) = \begin{bmatrix} f_1(x_p, y_p) \\ f_2(x_p, y_p) \end{bmatrix}$$

Step 2. Evoluate the Jacobian matrix

$$J(P_k) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

Step 3,

$$J(P_k) \cdot \Delta P = -f(P_k) \Rightarrow \Delta P = -J(P_k) \cdot f(P_k)$$

Step 4.

$$P_{k+1} = P_k + \Delta P$$

Ex

Consider the nonlineer system

$$x^2 - 2x - y + 0,5 = 0$$
$$x^2 + 4y^2 - 4 = 0$$

find the solution by NR method.

$(x_0, y_0) = (2, 0,25)$

Step 1 = $f(x,y) = \begin{bmatrix} x^2 - 2x - y + 0,5 \\ x^2 + 4y^2 - 4 \end{bmatrix} \Rightarrow f(2, 025)$

$$= \begin{bmatrix} 2 - 2.2 - 0,25 + 0,5 \\ 2^2 + 4.(0,25)^2 - 4 \end{bmatrix} = \begin{bmatrix} 0,25 \\ 0,25 \end{bmatrix}$$

Step 2 = $j(x,y) = \begin{bmatrix} 2x - 2 & -1 \\ 2x & 8y \end{bmatrix}$

$$J(2, 0,25) = \begin{bmatrix} 2.2 - 2 & -1 \\ 2.2 & 8.(0,25) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$$

Step 3 = $\begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \end{bmatrix} = - \begin{bmatrix} 0,25 \\ 0,25 \end{bmatrix}$

$$\begin{bmatrix} \Delta P_1 \\ \Delta P_2 \end{bmatrix} = - \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0,25 \\ 0,25 \end{bmatrix} = - \frac{1}{4+4} \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 0,25 \\ 0,25 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{-1}{8} \begin{bmatrix} 2.(0,25) + 1(0,25) \\ (-4)(0,25) + 2.(0,25) \end{bmatrix}$$

$$\begin{bmatrix} \Delta P_1 \\ \Delta P_2 \end{bmatrix} = \begin{bmatrix} -0,0938 \\ 0,0625 \end{bmatrix}$$

Step 4

$$R = R^0 + \Delta P \Longrightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2{,}00 \\ 0{,}25 \end{bmatrix} + \begin{bmatrix} -0{,}0938 \\ 0{,}0625 \end{bmatrix} = \begin{bmatrix} 1{,}9062 \\ 0{,}3125 \end{bmatrix}$$

$$\left.\begin{matrix} x_1 = 1{,}9062 \\ y_1 = 0{,}3125 \end{matrix}\right\} \quad x_{k+1} - x_k < \varepsilon \quad (\varepsilon = 0{,}01)$$

$$|1{,}9062 - 2| = 0{,}09 \not< \varepsilon$$

$$|0{,}3125 - 0{,}25| = 0{,}0625 \not< \varepsilon$$

$$|y_{k+1} - y_k| < \varepsilon$$

2. iteration

Step 1

$$f(x_1, y_1) = f(1{,}9062,\ 0{,}3125) =$$

Step 2

$$j(1{,}9062,\ 03125)$$

Step 3

$$\Delta P = \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \end{bmatrix}$$

Step 4

$$R_2 = R_1 + \Delta P_2 = \begin{bmatrix} 1{,}900691 \\ 0{,}311213 \end{bmatrix}$$

If $\varepsilon < 0{,}01$, you can get the solution

$$\# \quad R_3 \neq R_2 + \Delta P_3 = \begin{bmatrix} 1{,}90067 \\ 0{,}31121 \end{bmatrix} \qquad \varepsilon \cong 0{,}0001$$

Ex

$x^2+y^2=4$ find the solution by NR method.

$y+e^x=1$

$(x^0, y^0)=(1,-2)$

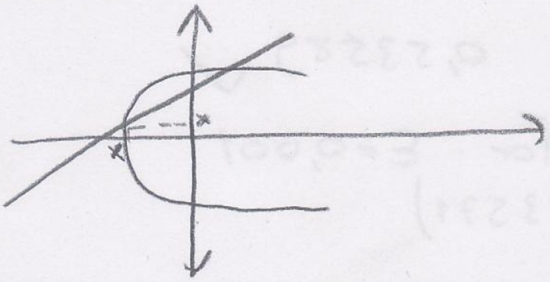
Clue:

$\neq 1$ $x_1 = x_0 + \Delta P_1 = 1+0{,}00426$
$y_1 = y_0 + \Delta P_2 = -2-0{,}02985$ } exercise at home

Ex

$y^2-4x-4=0$
$2y-x-2=0$ find the solution by NR

find the solution by NR plot the graphics



$(x^0, y^0)=(-1,1)$

Sol

1/ $f(x,y) = f(-1,1) = \begin{bmatrix} 1+4-4 \\ 2+1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Step 2/ $j(x,y) = \begin{bmatrix} -4 & 2y \\ -1 & 2 \end{bmatrix} \Rightarrow j(-1,1) = \begin{bmatrix} -4 & 2 \\ -1 & 2 \end{bmatrix}$

Step 3/ $\Delta p = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$= \dfrac{-1}{(-4)(2)-(2)(-1)} \begin{bmatrix} 2 & 2 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & -2 \\ 1 & +4 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$

Determinant

$$\Delta p = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \frac{4}{6} \\ -\frac{5}{6} \end{bmatrix}$$

Step 4

$$\begin{bmatrix} x^1 \\ y^1 \end{bmatrix} = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -1 \\ 0,5 \end{bmatrix}$$

$\neq$	Δx	Δy	x	y
0				
1	0	−0,5	−1	0,5
2	0,07142	0,03571	−0,92857	0,53571
3	0,000368	0,000184	−0,9282	0,53589 ✓

for $\varepsilon = 0,001$

roots $= (x, y) = (-0,92857 , 0,53571)$

x_k = root