→ Divided Differences

$P_n(x)$, $f(x)$ at $x_0, x_1, x_2 \ldots x_n$

$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots$

$a_0 = P(x_0) \cong f(x_0)$

$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) \cong f(x_1)$

① $a_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

• $f[x_i] = f(x_i)$ → bölünmüş farkı gösterir.

• $f[x_i, x_{i+1}] = \dfrac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$ → first divided difference

② $f[x_i, x_{i+1}, x_{i+2}] = \dfrac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$ → second divided difference

③ $f[x_i, x_{i+1} \ldots x_{i+k}] = \dfrac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots x_{i+k-1}]}{x_{i+k} - x_i}$

ⓝ n'th divided differance

$f[x_0, x_1, x_2 \ldots x_n] = \dfrac{f[x_1, x_2, \ldots x_n] - f[x_0, x_1 \ldots x_{n-1}]}{x_n - x_0}$

$P_n(x) = F[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2] \cdot (x-x_0)(x-x_1) + \ldots$

$P_n(x) = \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x-x_0)(x-x_1) \ldots (x-x_{k-1})$

x	$f(x)$	1. d.d (divided difference)	2. d.d (2. divided difference)
x_0	$f(x_0)$	$f(x_0,x_1) = \dfrac{f(x_1)-f(x_0)}{x_1-x_0}$	$f(x_0,x_1,x_2) = \dfrac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$
x_1	$f(x_1)$	$f(x_1,x_2) = \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$	$f(x_1,x_2,x_3) = \dfrac{f(x_2,x_3)-f(x_1,x_2)}{x_3-x_1}$
x_2	$f(x_2)$	$f(x_2,x_3)$	⋮
x_3	$f(x_3)$	$f(x_3,x_4)$	
x_4	$f(x_4)$	$f(x_4,x_5)$	
⋮		⋮	

Ex

x	$f(x)$
1	0,7651977
1,3	0,6200860
1,6	0,4554022
1,9	0,2818186
2,2	0,110323

find $f(1,5) = ?$

by Newton-Interpolatin Poly. differences

(Newton Interpolating Polynomial)

i	x_i	$f(x_i)$	$f(x_{i-1}, x_i)$	$f(x_{i-2}, x_{i-1}, x_i)$
0	1	0,7651977	$\to$ −0,4837057	−
1	1,3	0,6200860	$\to$ −0,5489460	$\to$ −0,108739
2	1,6	0,4554022	$\to$ −0,5786120	$\to$ −0,0694433
3	1,9	0,2818186	$\to$ −0,5715210	$\to$ 0,0118183
4	2,2	0,1103623		
			a_1	a_2

1. dd (a) $f[x_0,x_1] = \dfrac{f(x_1)+f(x_0)}{x_1-x_0} = \dfrac{0,620086-0,7651977}{1.3-1}$

$= -0,4837057$

(b) $f[x_1,x_2] = \dfrac{0,4554022-0,6200860}{1.6-1.3} = -0,5489460$

2. dd $f[x_0,x_1,x_2] = \dfrac{\overset{b}{f[x_1,x_2]} - \overset{a}{f[x_0,x_1]}}{x_2-x_0} = \dfrac{-0,5489460-(-0,483705)}{1.6-1}$

$= -0,1087339$

$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$ $\quad$ $f[x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}, x_i]$

$\to 0,0658784$ a_3 $\quad\longrightarrow 0,0018251$ (a_4)

$\to 0,0680685$

3.d.d

$$f[x_0,x_1,x_2,x_3] = \frac{\overset{b_1}{f(x_1,x_2,x_3)} - \overset{a_1}{f(x_0,x_1,x_2)}}{x_3 - x_0}$$

$$= \frac{-0,0494433 - (-0,1087339)}{1.9-1}$$

$$= 0,0,0658784$$

$$f[x_1,x_2,x_3,x_4] = \frac{0,118183 - (0,0494433)}{2.2-1,3} = 0,0680685$$

4.dd

$$f[x_0,x_1,x_2,x_3,x_4) = \frac{\overset{b_2}{0,0680685} - \overset{a_2}{(0,0658784)}}{2.2-1}$$

$$= 0,0018221$$

$$P_n(x) = P_4(x) = 0,7651997 - 0,4837057(x-1,0) - 0,1087339(x-1).(x-1,3)$$

$$+ 0,0658784(x-1).(x-(1,3)(x-1,6)$$

$$+ 0,0018251(x-1).(x-1,3)(x-1,6)(x-1,9)$$

$$P_n(1,5) \cong f(1,5) = 0,5118200$$

Newton Divided Differences Approximations

→ forward Differences → üstteki değerleri kullan

<u>Note!</u> When the points are arranged with equal spacing consecutively, Newton's divided difference formula can be expressed in a simplified form,

$$h = x_{i+1} - x_i \;,\quad x_1 = x_0 + h,\quad x_2 = x_1 + h \ldots x_n = x_{n-1} + h$$

$$x_2 = x_0 + 2h \ldots\; x_s = x_0 + sh$$

$$P_n(x) = P_n(x_0 + sh) = f(x_s) + sh\, f[x_0, x_1] + s(s-1).h^2.f[x_0, x_1, x_2] + s.(s-1)(s-n+1).h^n.f[x_0, x_1 \ldots x_k]$$

↳ general formula

$h \to$ costant (sabit)

$$\binom{s}{k} = c(s,k) = \frac{s!}{k!.(s-k)!}$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}\left[f(x_1) - f(x_0)\right] = \frac{1}{h}\Delta f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{1}{2h}\left[\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right] = \frac{1}{2h^2}\,\Delta^2 f(x_0)$$

↳ Aitken's

$$\Delta f_i = f_{i+1} - f_i$$

<u>General formula</u>

$$f(x_0, x_1, x_2, \ldots x_k) = \frac{1}{k!.h^k}.\Delta^k.f(x_0)$$

↳ Newton forward Difference formula

→ Backward Differences: Altaki değerleri kullan

If the interdating nodes are recorded from last to first as $x_n, x_{n-1}, \dots, x_0$ we can write the interpolatory formula as

$$P_n(x) = f[x_n] + f[x_n, x_{n-1}](x-x_n) + f[x_n, x_{n-1}, x_{n-2}](x-x_n)(x-x_{n-1})$$

if the points are equally spaced with

$x = x_n + sh$, $x = x_i + (s+n-i)h$ then

$$P_n(x) = P_n(x_n + sh)$$
$$= f[x_n] + sh f[x_n, x_{n-1}] + s(s+1)h^2 f[x_n, x_{n-1}, x_{n-2}] + \dots$$

Newton backward-differences formula

$$P_n(x) = f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n).$$

Ex find the table at divided differences.

x	$f(x)$
3	-459
8	-18864
5	-3105
1	-13
7	-11263

Question

x_i	$f(x_i)$	$f(x_i, x_{i+1})$	$f(x_i, x_{i+1}, x_{i+2})$	$f(x_i, x_{i+1}, x_{i+2}, x_{i+3})$	
3	-459				
		-3681			
8	-18864		-786		
		-5253		-73	
5	-3105		-640		-4
		-773		-89	
1	-13		-551		
		-1875			
7	-11263				

=2,2 since this choise makes the earlist possible use of data is closest to $x = 1,1$

$P_4(1,1) = P(1,0 + \frac{1}{3}.0,3) \rightarrow P(x_0 + sh)$ Sınavda S bulunacak

Ex

$f(1,1) = ?$ → forward ile çözdü.

$P_4(1,1) = P_4(x_0+sh)$ $\qquad h = 1,3-1 = 1,6-1,3 = 1,9-1,6$

$1,1 = 1,0 + s \cdot 0,3$ $\qquad h = 1,3-1 = x_1 - x_0 = 0,3$

$$s = \frac{0,1}{0,3} = \frac{1}{3}$$

$\boxed{\text{Tablo 2. sayfadaki soruda}}$

$$P_4\left(1+\frac{1}{3}(0,3)\right) = 0,7651977 + \frac{1}{3}(0,3)(-0,4837057)$$

$$+\frac{1}{3}\left(-\frac{2}{3}\right)(0,3)^2 \cdot (-0,1087339)$$

$$+\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right) \cdot (0,3)^3 \cdot (0,0658784)$$

$$+\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right) \cdot (0,3)^4 \cdot (0,0018251)$$

$$= \underline{0,7196460}$$

Ex

$f(2,0) = ?$ $\qquad$ backword → sondan başa git

$h = 1,3 - 1 = 0,3$

$P_4(2,0) = P_4(x_n + sh)$

$2.0 = 2.2 + s \cdot 0,3 \Rightarrow s = \frac{-2}{3}$

$$= 0,1103623 - \frac{2}{3}(0,3)(-0,5715210) - \frac{2}{3}\left(\frac{1}{3}\right)(0,3)^2$$

$$\cdot (0,118183) - \frac{2}{3}\left(\underbrace{\frac{1}{3}}_{s+1}\right)\cdot\left(\underbrace{\frac{4}{3}}_{s+2}\right)(0,3)^3\ (0,0680685)$$

$$\frac{-2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)(0,3)^4 \cdot (0,0018251) = \underline{0,2238754}$$