$$\frac{\left(y+\sqrt{x^2-y^2}\right)dx - x\,dy}{x} = \frac{0}{x}$$

$$\left(\frac{y}{x}+\sqrt{1-\left(\frac{y}{x}\right)^2}\right)dx - dy = 0 \qquad \frac{y}{x}=u \leadsto y=ux,\quad dy = u\,dx + x\,du$$

$$\left(u+\sqrt{1-u^2}\right)dx - (u\,dx + x\,du) = 0$$

$$\left(\cancel{u}+\sqrt{1-u^2}-\cancel{u}\right)dx = x\,du \rightarrow \sqrt{1-u^2}\,dx = x\,du \rightarrow \int\frac{dx}{x} = \int\frac{du}{\sqrt{1-u^2}}$$

$$\ln|x| = \operatorname{Arcsin} u + c$$

$$\ln|x| = \operatorname{Arcsin}\frac{y}{x} + c$$

4°) Homogen veya değişkenlerine ayrılabilen hale getirilebilen dif. denk.

$$(a_1x+b_1y+c_1)dx + (a_2x+b_2y+c_2)dy = 0$$

i) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \Rightarrow \left.\begin{matrix} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{matrix}\right\}$ $\begin{matrix} x=h \\ y=k \end{matrix}$ gibi bir çözümü vardır.

$\left.\begin{matrix} x=\bar{x}+h \\ y=\bar{y}+k \end{matrix}\right\}$ $\left.\begin{matrix} dx=d\bar{x} \\ dy=d\bar{y} \end{matrix}\right\}$ dönüşümü ile denklem $(a_1\bar{x}+b_1\bar{y})d\bar{x}+(a_2\bar{x}+b_2\bar{y})d\bar{y}=0$ homogen dif. denk. dönüştürülerek çözülür.

$$(x+y-3)dx + (-x+y+1)dy = 0$$

$\begin{matrix} x+y-3=0 \\ -x+y+1=0 \end{matrix} \leadsto \left.\begin{matrix} x+y=3 \\ -x+y=-1 \end{matrix}\right\}$ $\begin{matrix} x=2 \\ y=1 \end{matrix}$ $\left.\begin{matrix} x=\bar{x}+2 \\ y=\bar{y}+1 \end{matrix}\right\}$ $\begin{matrix} dx=d\bar{x} \\ dy=d\bar{y} \end{matrix}$

$$(\bar{x}+\cancel{2}+\bar{y}+\cancel{1}-\cancel{3})d\bar{x} + (-\bar{x}-\cancel{2}+\bar{y}+\cancel{1}+\cancel{1})d\bar{y} = 0$$

$$\frac{(\bar{x}+\bar{y})d\bar{x}+(\bar{y}-\bar{x})d\bar{y}}{\bar{x}} = \frac{0}{\bar{x}} \rightarrow \left(1+\frac{\bar{y}}{\bar{x}}\right)d\bar{x} + \left(\frac{\bar{y}}{\bar{x}}-1\right)d\bar{y} = 0$$

$\frac{\bar{y}}{\bar{x}}=u \qquad \bar{y}=u.\bar{x},\quad d\bar{y}=u\,d\bar{x}+\bar{x}\,du \rightarrow (1+u)d\bar{x} + (u-1)(u\,d\bar{x}+\bar{x}\,du) = 0$

$$(1+\cancel{u}+u^2-\cancel{u})d\bar{x} + \bar{x}(-1+u)du = 0$$

$$(1+u^2)d\bar{x} = \bar{x}(1-u)du$$

$$\frac{d\bar{x}}{\bar{x}} = \frac{1-u}{1+u^2}.du$$

$$\int\frac{d\bar{x}}{\bar{x}} = \int\frac{1}{1+u^2}du - \frac{1}{2}\int\frac{2u}{1+u^2}.du$$

$$\ln\bar{x} = \operatorname{Arctan} u - \frac{1}{2}\ln|1+u^2| + c$$

$$\ln\bar{x} = \operatorname{Arctan}\frac{\bar{y}}{\bar{x}} - \frac{1}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| + c \rightarrow \begin{matrix} \bar{x}=x-2 \\ \bar{y}=y-1 \end{matrix}$$

$$\ln|x-2| = \operatorname{Arctan}\left(\frac{y-1}{x-2}\right) - \frac{1}{2}\ln\left|1+\left(\frac{y-1}{x-2}\right)^2\right| + c$$

ii) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \rightarrow \left.\begin{matrix} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{matrix}\right\}$ $\begin{matrix} x=h \\ y=k \end{matrix}$ gibi çözüm yok!

$u = a_1x + b_1y$

$du = a_1dx + b_1dy \rightsquigarrow dy = \dfrac{du - a_1dx}{b_1}$ dönüşümü ile değişkenlerine ayrılabilir hale getirilerek çözülür.

Örn: $(x+y+1)dx + (2x+2y+1)dy = 0$

$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$ $\quad u = x+y$

$du = dx+dy \rightsquigarrow dy = du - dx$

$(u+1)dx + (2u+1).(du-dx) = 0$

$(u+1-2u-1)dx + (2u+1).du = 0$

$-u\,dx + (2u+1)du = 0 \rightarrow dx = \dfrac{(2u+1)}{u}du$

$\int dx = \int 2du + \int \frac{1}{u}.du$

$x = 2u + \ln|u| + c \rightsquigarrow x = 2(x+y) + \ln|x+y| + c$

Ödev: $(2x+4y+2)dx - (4x+8y-3)dy = 0$

$\begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 0$ $\quad 2x+4y = u$

$2dx + 4dy = du \rightarrow dy = \dfrac{du - 2dx}{4}$

$(u+2)dx - (2u-3)\left(\dfrac{du-2dx}{4}\right) = 0$

$\left(u+2+u-\frac{3}{2}\right)dx + \left(-\frac{1}{2}u + \frac{3}{4}\right)du = 0$

$\left(2u+\frac{1}{2}\right)dx = \left(\frac{1}{2}u - \frac{3}{4}\right)du$

$(8u+2)dx = \left(u - \frac{3}{4}\right)du \rightarrow dx = \left(\dfrac{u-\frac{3}{4}}{8u+2}\right)du$

$dx = \dfrac{4u-3}{32u+8}$

5) Tam Diferansiyel Denklemler

Hatırlatma: Toplam Diferansiyel

$f(x,y)=c$ ifadesinin toplam diferansiyeli

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0 \quad \text{idi.}$$

Örneğin; $2x^3y^2=c$ 'nin toplam diferansiyeli

$$6x^2y^2dx+4x^3ydy=0 \quad \text{'dır.}$$

—o—

$P(x,y)dx+Q(x,y)dy=0$ diferansiyel denklemi eğer $u(x,y)=c$ gibi bir ifadenin toplam diferansiyeli ise yani

$\frac{\partial u}{\partial x}.dx+\frac{\partial u}{\partial y}.dy=0$ şeklinde ise $P(x,y).dx+Q(x,y)dy=0$ tam dif. denk.

Hatırlatma: $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$ idi.

—o—

Eğer $P(x,y)dx+Q(x,y)dy=0$ tam dif. ise $\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0$ şeklinde idi.

Bu durumda $P(x,y)=\frac{\partial u}{\partial x}$; $Q(x,y)=\frac{\partial u}{\partial y}$ diye kabul edebiliriz.

Ayrıca $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}\rightarrow\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

Tam Dif. Olma Şartı: $\boxed{\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}}$ dır.

$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0\rightsquigarrow u(x,y)=c$

$P(x,y)dx+Q(x,y)dy=0 \rightsquigarrow \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ şartını sağlarsa tam dif. denklemdir.

$$\left.\begin{array}{l}\bullet\ \frac{\partial u}{\partial x}=P(x,y)\\ \bullet\bullet\ \frac{\partial u}{\partial y}=Q(x,y)\end{array}\right\} \overset{??}{u(x,y)=c}$$

$\bullet\ \frac{\partial u}{\partial x}=P(x,y)\Rightarrow\int\partial u=\int P(x,y)dx+R(y)$

$$u=\int P(x,y)dx+R(y)$$

$\bullet\bullet\ \frac{\partial u}{\partial y}=Q(x,y)\Rightarrow\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left[\int P(x,y)dx\right]+\frac{dR(y)}{dy}=Q(x,y)$

$\frac{dR(y)}{dy}$ çekilerek integre edilir. $R(y)$ bulunur.

$u=\int P(x,y)dx+R(y)$ de yerine yazılır.

Sonuç: $u(x,y)=c$ haline getirilerek soru çözülür.

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$$

Soru: $\underbrace{\left(x^2+\frac{y^2}{x}\right)}_{P} dx + \underbrace{2y\ln x}_{Q}\,dy = 0$

$$\frac{\partial P}{\partial y} = 0+\frac{2y}{x}$$
$$\frac{\partial Q}{\partial x} = 2y\cdot\frac{1}{x}$$
= Tam dif.

• $\frac{\partial u}{\partial x} = x^2+\frac{y^2}{x}$

•• $\frac{\partial u}{\partial y} = 2y\ln x$

?? $u(x,y)=c$

• $\frac{\partial u}{\partial x} = x^2+\frac{y^2}{x}$ ise $\rightarrow$ $u=\int\left(x^2+\frac{y^2}{x}\right)dx + R(y)$

$u=\frac{x^3}{3}+y^2\cdot\ln x + R(y)$

•• $\frac{\partial u}{\partial y} = 2y\ln x$ ise $\Rightarrow$ $\frac{\partial u}{\partial y} = 0+\cancel{2y\ln x}+\frac{dR(y)}{dy} = \cancel{2y\ln x}$

$\frac{dR(y)}{dy}=0 \rightarrow R(y)=k$

$u=\frac{x^3}{3}+y^2\ln x + R(y)$ ise $\rightarrow$ $u=\frac{x^3}{3}+y^2\ln x+K$; $u(x,y)=c$ ise

$\frac{x^3}{3}+y^2\ln x+K=C \rightarrow \boxed{\frac{x^3}{3}+y^2\ln x = C-K}$ $\rightarrow C_1$

$\frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy=0$

Soru: $\underbrace{\frac{y}{x}}_{P}.dx + \underbrace{(y^3+\ln x)}_{Q}dy = 0$

$$\frac{\partial P}{\partial y} = \frac{1}{x}$$
$$\frac{\partial Q}{\partial x} = 0+\frac{1}{x}$$
= Tam dif.

• $\frac{\partial u}{\partial x} = \frac{y}{x}$

•• $\frac{\partial u}{\partial y} = y^3+\ln x$

?? $u(x,y)=c$

• $\frac{\partial u}{\partial x} = \frac{y}{x}$ ise $\Rightarrow$ $u=\int\frac{y}{x}.dx + R(y)$

$u=y\ln x + R(y)$

•• $\frac{\partial u}{\partial y} = y^3+\ln x$ ise $\Rightarrow$ $\frac{\partial u}{\partial y} = \cancel{\ln x}+\frac{dR(y)}{dy} = y^3+\cancel{\ln x}$

$\frac{dR(y)}{dy} = y^3 \rightarrow \int dR(y) = \int y^3 dy$

$R(y)=\frac{y^4}{4}+K$

$u=y\ln x+R(y)$ ise

$u=y\ln x+\frac{y^4}{4}+K$

$u(x,y)=C \rightarrow y\ln x+\frac{y^4}{4}+K=C \rightarrow \boxed{y\ln x+\frac{y^4}{4} = C-K}$ $\rightarrow C_1$