4-) Find a power series expansion including the first five nonzero terms about the point $x_0=0$ of the general solution to the differential equation

$$(3-x^2)y''-3xy'-y=0.$$

$$y=\sum_{n=0}^{\infty} c_n x^n, \quad y'=\sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\underbrace{\sum_{n=2}^{\infty} 3n(n-1)c_n x^{n-2}}_{\sum_{n=0}^{\infty} 3(n+2)(n+1)c_{n+2}x^n} + \sum_{n=2}^{\infty} -n(n-1)c_n x^n + \sum_{n=1}^{\infty} -3n c_n x^n + \sum_{n=0}^{\infty} -c_n x^n = 0$$

$$3\cdot2\cdot1\, c_2 + 3\cdot3\cdot2\, c_3 x - 3c_1 x - c_0 - c_1 x + \sum_{n=2}^{\infty}\left[3(n+1)(n+2)c_{n+2} + (-n(n-1)-3n-1)c_n\right]x^n = 0$$

$$6c_2 - c_0 = 0 \to \boxed{c_2=\frac{1}{6}c_0} \; ; \quad 18c_3 - 4c_1 = 0 \to \boxed{c_3=\frac{2}{9}c_1}$$

$$3(n+1)(n+2)c_{n+2} - (n^2-n+3n+1)c_n = 0, \quad n\geq 2$$

$$c_{n+2} = \frac{n^2+2n+1}{3(n+1)(n+2)}c_n \to c_{n+2} = \frac{(n+1)^2}{3(n+1)(n+2)}c_n$$

$$\Rightarrow \boxed{c_{n+2} = \frac{n+1}{3(n+2)}c_n, \quad n\geq 2}$$

$$n=2; \quad c_4 = \frac{3}{3\cdot4}c_2 = \frac{3}{3\cdot4}\cdot\frac{1}{6}c_0 \to \boxed{c_4=\frac{1}{24}c_0}$$

$$n=3; \quad c_5 = \frac{4}{3\cdot5}c_3 = \frac{4}{3\cdot5}\cdot\frac{2}{9}c_1 \to \boxed{c_5=\frac{8}{135}c_1}$$

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$$

$$y = c_0 + c_1x + \frac{1}{6}c_0x^2 + \frac{2}{9}c_1x^3 + \frac{1}{24}c_0x^4 + \frac{8}{135}c_1x^5 + \cdots$$

$$y = c_0\left(1+\frac{1}{6}x^2+\frac{1}{24}x^4+\cdots\right) + c_1\left(x+\frac{2}{9}x^3+\cdots\right)$$

YTU – Faculty of Arts and Sciences Final Exam Questions and Solutions Sheet			Score Table				
			1	2	3	4	TOTAL
Name-Surname							
Number		Group No					
Department				Date	03.01. 2017		
Course	MAT2411 Differential Equations		Duration	90 mins.	Room		
Lecturer			Signature				

YÖK nun 2547 sayılı Kanunun *Öğrenci Disiplin Yönetmeliğinin* 9. Maddesi olan *"Sınavlarda kopya yapmak ve yaptırmak veya buna teşebbüs etmek"* fiili işleyenler bir veya iki yarıyıl uzaklaştırma cezası alırlar.

1- a) Find the differential equation of the family of curves $y=\dfrac{1}{c+x^2}$, $(c\in\mathbb{R}^+)$.

$$y' = -(c+x^2)^{-2}\cdot 2x$$

$$y' = -\frac{2x}{(c+x^2)^2} \to y' = \frac{-2x}{\left(\frac{1}{y}\right)^2} \Rightarrow y' = -2xy^2$$

b) Solve the initial value problem $(e^{-2y}-y)\cos x\,dy = e^y\sin(2x)dx$, $y(0)=0$.

$$\frac{e^{-2y}-y}{e^y}dy = \frac{\sin 2x}{\cos x}dx$$

$$(e^{-3y}-ye^{-y})dy = \frac{2\sin x\cos x}{\cos x}dx$$

$$\int(e^{-3y}-ye^{-y})dy = \int 2\sin x\,dx$$

$$-\frac{1}{3}e^{-3y} + (1+y)e^{-y} = -2\cos x + C$$

$$\int ye^{-y}dy = -ye^{-y} + \int e^{-y}dy$$

$$\left[\begin{array}{ll} u=y, & dv=e^{-y}dy \\ du=dy & v=-e^{-y} \end{array}\right]$$

$$= -ye^{-y} - e^{-y} + c$$

$$= -(1+y)e^{-y} + c$$

$$y(0)=0 \to -\frac{1}{3}+1 = -2+C$$

$$C = \frac{8}{3}$$

$$\Rightarrow -\frac{1}{3}e^{-3y} + (1+y)e^{-y} = -2\cos x + \frac{8}{3} \;//$$

2-a) Evaluate the integral $\int_0^\infty e^{-st}(1+t\,\sin t)dt$ by using Laplace transform definition and properties of Laplace transform.

$$\int_0^\infty e^{-st}(1+t\sin t)\,dt = \mathcal{L}\{1+t\sin t\}$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{t\sin t\}$$

$$= \frac{1}{s} + (-1)\frac{d}{ds}\mathcal{L}\{\sin t\}$$

$$= \frac{1}{s} - \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \underline{\frac{1}{s} + \frac{2s}{(s^2+1)^2}}$$

b) Using Laplace transform, find the solution of the initial value problem

$$y''+2y'+5y=0,\quad y(0)=2,\ y'(0)=-1.$$

$\mathcal{L}\{y\} = Y(s)$; $\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s + 1$

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{0\}$

$s^2Y(s) - 2s + 1 + 2sY(s) - 4 + 5Y(s) = 0$

$(s^2+2s+5)Y(s) = 2s+3 \quad \to \quad Y(s) = \dfrac{2s+3}{s^2+2s+5}$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2s+2+1}{(s+1)^2+4}\right\}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$\underline{y(t) = 2e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t}$$

3-) Solve the system of differential equations

$$\frac{dy}{dx} = 2y - z + 1 \qquad (1)$$

$$\frac{dz}{dx} = y + e^{2x} \qquad (2)$$

using *Elimination Method.*

$(2)' \to \dfrac{d^2z}{dx^2} = \dfrac{dy}{dx} + 2e^{2x} \qquad (3)$

$(1) - 2(2) \to \dfrac{dy}{dx} - 2\dfrac{dz}{dx} = -z + 1 - 2e^{2x}$

$\underline{\dfrac{dy}{dx} = 2\dfrac{dz}{dx} - z + 1 - 2e^{2x}}$

(3) de yazılırsa $\dfrac{d^2z}{dx^2} = 2\dfrac{dz}{dx} - z + 1 - \cancel{2e^{2x}} + \cancel{2e^{2x}}$

$$\boxed{\frac{d^2z}{dx^2} - 2\frac{dz}{dx} + z = 1}$$

$r^2 - 2r + 1 = 0 \to (r-1)^2 = 0$

$r = 1$ (2 katlı)

$z_h = (C_1 + C_2x)e^x$

$z_ö = A, \quad z_ö' = z_ö'' = 0$

$A = 1$

$\underline{z_ö = 1}$

$z = z_h + z_ö$

$$\boxed{z = (C_1 + C_2x)e^x + 1}$$

(2) denkleminden

$y = \dfrac{dz}{dx} - e^{2x}$

$y = C_2e^x + (C_1 + C_2x)e^x - e^{2x}$

$$\boxed{y = (C_1 + C_2 + C_2x)e^x - e^{2x}}$$

Sistemin g.ç.:

$$\begin{cases} y = (C_1 + C_2 + C_2x)e^x - e^{2x} \\ z = (C_1 + C_2x)e^x + 1 \end{cases}$$