İntegrasyon Teknikleri

① Yerine Koyma (Değişken Değiştirme) Kuralı

Eğer $u=g(x)$, değer kümesi I aralığı olan türevlenebilir bir fonk. ve f de I üzerinde sürekli ise:

①

$$\int f(g(x)).g'(x)\,dx = \int f(u)\,du \text{ dur.}$$

$$\left(u=g(x) \Rightarrow du = g'(x)\,dx\right)$$

②

$$\int_a^b f(g(x)).g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$\Big(u=g(x) \Rightarrow du = g'(x)\,dx$

Eski sınır		Yeni sınır
$x=a$	$\longrightarrow$	$u=g(a)$
$x=b$	$\longrightarrow$	$u=g(b)$ $\Big)$

⊛ $I=\int (x^3+x)^5.(3x^2+1)\,dx = ?$ $\qquad x^3+x=u \Rightarrow (3x^2+1)\,dx=du$

$$I=\int \underbrace{(x^3+x)^5}_{u^5}.\underbrace{(3x^2+1)\,dx}_{du} = \int u^5\,du = \frac{u^6}{6}+C = \frac{(x^3+x)^6}{6}+C$$

⊛ $\int x^2 \operatorname{Sin}x^3\,dx = ?$ $\qquad x^3=u \Rightarrow 3x^2dx=du \Rightarrow x^2dx=\frac{du}{3}$

$$\int \underbrace{\operatorname{Sin}x^3}_{\operatorname{Sin}u}.\underbrace{x^2dx}_{\frac{du}{3}} = \frac{1}{3}\int \operatorname{Sin}u\,du = -\frac{\operatorname{Cos}u}{3}+C = -\frac{\operatorname{Cos}x^3}{3}+C$$

⊛ $$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \underbrace{\sqrt{u}}_{u^{1/2}}\,du = \frac{u^{3/2}}{\frac{3}{2}}\Bigg|_0^2 = \frac{2}{3}.2\sqrt{2} = \frac{4\sqrt{2}}{3}$$

$x^3+1=u$
$3x^2dx=du$
$x=1 \Rightarrow u=2$
$x=-1 \Rightarrow u=0$

⊛ $\int x\sqrt{2x+1}\,dx = \int \frac{u-1}{2}\cdot\sqrt{u}\,\frac{du}{2} = \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du$

$\left(\begin{array}{l} 2x+1=u \\ 2dx=du \end{array}\right\} \begin{array}{l} dx=\frac{du}{2} \\ x=\frac{u-1}{2} \end{array}\Bigg)$

$= \frac{1}{4}\left(\frac{u^{5/2}}{\frac{5}{2}} - \frac{u^{3/2}}{\frac{3}{2}}\right)+c$

$= \frac{1}{10}\cdot(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2}+c$

⊛ $\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx = \int_1^3 2\cos u\,du = 2\sin u\Big|_1^3 = 2\sin 3 - 2\sin 1$

$\sqrt{x+1}=u \quad \frac{dx}{2\sqrt{x+1}}=du$

$x=8 \Rightarrow u=3$

$x=0 \Rightarrow u=1$

⊛ $\int \frac{2x}{\sqrt[3]{x^2+1}}\,dx = \int \frac{du}{\sqrt[3]{u}} = \int u^{-1/3}\,du = \frac{u^{2/3}}{\frac{2}{3}}+c = \frac{3}{2}\,(x^2+1)^{2/3}+c$

$x^2+1=u \rightarrow 2x\,dx=du$

Tanx, Cotx, Secx, Cosecx İntegralleri

⊛ $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\ln|u|+c = -\ln|\cos x|+c$

$\cos x=u$

$\downarrow$

$-\sin x\,dx=du$

⊛ $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u|+c = \ln|\sin x|+c$

$\sin x=u \Rightarrow \cos x\,dx=du$

⊛ $\int \sec x\,dx = \int \sec x\cdot\frac{(\sec x+\tan x)}{\sec x+\tan x} = \int \frac{\sec^2 x+\sec x\cdot\tan x}{\sec x+\tan x}\,dx$

$\sec x+\tan x=u$

$\Downarrow$

$(\sec x\tan x+\sec^2 x)\,dx=du$

$= \int \frac{du}{u} = \ln|u|+c$

$= \ln|\sec x+\tan x|+c$

(*) $\int \operatorname{Cosec} x \, dx = \int \operatorname{Cosec} x \cdot \frac{(\operatorname{Cosec} x + \operatorname{Cot} x)}{\operatorname{Cosec} x + \operatorname{Cot} x} dx = \int \frac{\operatorname{Cosec}^2 x + \operatorname{Cot} x \operatorname{Cosec} x}{\operatorname{Cosec} x + \operatorname{Cot} x} dx$

$\operatorname{Cosec} x + \operatorname{Cot} x = u$
$\downarrow$
$-(\operatorname{Cosec} x \operatorname{Cot} x + \operatorname{Cosec}^2 x) dx = du$

$= -\int \frac{du}{u} = -\ln|u| + C$
$= -\ln|\operatorname{Cot} x + \operatorname{Cosec} x| + C$

$\operatorname{Sin}^2 x$ ve $\operatorname{Cos}^2 x$ İntegralleri

$\operatorname{Sin}^2 x = \frac{1 - \operatorname{Cos} 2x}{2}$ ve $\operatorname{Cos}^2 x = \frac{1 + \operatorname{Cos} 2x}{2}$ özdeşlikleri kullanılarak çözülür.

(*) $\int \operatorname{Sin}^2 x \, dx = \int \frac{1 - \operatorname{Cos} 2x}{2} dx = \frac{x}{2} - \frac{\operatorname{Sin} 2x}{4} + C$

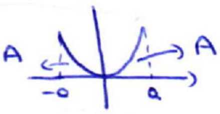
(*) $\int \operatorname{Cos}^2 x \, dx = \int \frac{1 + \operatorname{Cos} 2x}{2} dx = \frac{x}{2} + \frac{\operatorname{Sin} 2x}{4} + C$

Simetrik fonksiyonların Belirli İntegrali

$f(x)$, $[-a, a]$ da sürekli olsun.

a) f, çift fonksiyon ise : $\int_{-a}^{a} f(x) dx = 2 \int_{0}^{a} f(x) dx$ dir.

(Çünkü $f(x)$ y-eksenine göre simetriktir.



$\Rightarrow \int_{-a}^{a} f(x) dx = 2A$
$= 2 \int_{0}^{a} f \, dx$)

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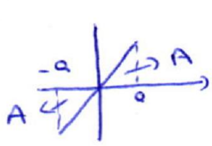
b) f, tek fonk. ise : $\int_{-a}^{a} f(x) dx = 0$ dır.

Çünkü $f(x)$ orjine göre simetriktir.



$\int_{-a}^{a} f(x) dx = A - A = \underline{\underline{0}}$

Ters Trigometrik fonksiyonları veren integraller

⊛ $\int \frac{dx}{\sqrt{1-x^2}} = \operatorname{ArcSin} x + c$

⊛ $\int \frac{dx}{1+x^2} = \operatorname{ArcTan} x + c$

⊛ $\int \frac{dx}{4+x^2} = \frac{1}{4}\int \frac{dx}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\operatorname{ArcTan} u + c = \frac{1}{2}\operatorname{ArcTan}\frac{x}{2} + c$

$\left(u = \frac{x}{2} \Rightarrow du = \frac{dx}{2}\right)$

KURAL:

$$\boxed{\star \int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{ArcTan}\frac{x}{a} + c}$$

⊛ $\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{dx}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \operatorname{ArcSin} u + c = \operatorname{ArcSin}\frac{x}{2} + c$

$\frac{x}{2} = u \to \frac{dx}{2} = du$

KURAL : $\boxed{\star \int \frac{dx}{\sqrt{a^2-x^2}} = \operatorname{ArcSin}\frac{x}{a} + c}$

⊛ $\int \frac{dx}{x^2+3} = \frac{1}{\sqrt{3}}\operatorname{ArcTan}\frac{x}{\sqrt{3}} + c$

⊛ $\int \frac{dx}{\sqrt{5-x^2}} = \operatorname{ArcSin}\frac{x}{\sqrt{5}} + c$

⊛ $\int \frac{dx}{x^2+8} = \frac{1}{2\sqrt{2}} \cdot \operatorname{ArcTan}\frac{x}{2\sqrt{2}} + c$

⊛ $\int \frac{dx}{\sqrt{16-x^2}} = \operatorname{ArcSin}\frac{x}{4} + c$

② Kısmi İntegrasyon

$u(x)$ ve $v(x)$ türevlenebilir fonksiyonlar olsun.

$$\frac{d}{dx}(u.v) = v.\frac{du}{dx} + u.\frac{dv}{dx}$$

$$\downarrow$$

$$\underbrace{\int \frac{d}{dx}(u.v)dx}_{} = \int v\frac{du}{dx}.dx + \int u.\frac{dv}{dx}$$

$$\downarrow$$

$$u.v = \int v du + \int u dv \Rightarrow \boxed{\int u dv = u.v - \int v du} \rightarrow$$ Kısmi İntegrasyon formülü

☆ Belirli integralde : $\boxed{\int_a^b u dv = u.v\Big|_a^b - \int_a^b v du}$

☆ u deme sıralaması : LAPTÜ

- Ü → Üstel fonk.
- T → Trigonometrik fonk.
- P → Polinom
- A → Ters Trig. fonk.
- L → Logaritmik fonk.

☆ $\boxed{\begin{array}{l} u \xrightarrow{\text{Türev}} du \\ dv \xrightarrow{\text{integral}} v \end{array}}$

⊛ $\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$

$x = u \rightarrow dx = du$

$e^x dx = dv \rightarrow e^x = v$

⊛ $\int \ln x\, dx = x\ln x - \int dx = x \ln x - x + C$

$\ln x = u$, $dx = dv$

$\downarrow$ $\downarrow$

$\frac{dx}{x} = du$ $x = v$

⊛ $\int x^2 \sin x\, dx = -x^2\cos x + 2\int x \cos x\, dx = -x^2\cos x + 2x\sin x - 2\int \sin x\, dx$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$

$x^2 = u$	$\sin x\, dx = dv$	$x = u$	$\cos x\, dx = dv$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$2x dx = du$	$-\cos x = v$	$dx = du$	$\sin x = v$

$$\int x \,\mathrm{Arctan}x \, dx = \frac{x^2}{2}\mathrm{Arctan}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

$\mathrm{Arctan}x = u \quad \downarrow \quad \frac{dx}{1+x^2} = du$

$x\,dx = dv \quad \downarrow \quad \frac{x^2}{2} = v$

$$= \frac{x^2}{2}\mathrm{Arctan}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\mathrm{Arctan}x - \frac{x}{2} + \frac{\mathrm{Arctan}x}{2} + C$$

$$\int \mathrm{ArcSin}x \, dx = x\,\mathrm{ArcSin}x - \int \frac{x}{\sqrt{1-x^2}}dx = x\,\mathrm{ArcSin}x + \frac{1}{2}\int\frac{dt}{\sqrt{t}}$$

$\mathrm{ArcSin}x = u \quad \downarrow \quad \frac{dx}{\sqrt{1-x^2}} = du$

$dx = dv \quad \downarrow \quad x = v$

$1-x^2 = t$

$-2x\,dx = dt$

$$= x\,\mathrm{ArcSin}x + \frac{1}{2}\cdot 2\sqrt{1-x^2} + C$$

$$= x\,\mathrm{ArcSin}x + \sqrt{1-x^2} + C$$

$$I = \int e^x \mathrm{Cos}x \, dx = e^x \mathrm{Cos}x + \int e^x \mathrm{Sin}x \, dx$$

$\mathrm{Cos}x = u \quad \downarrow \quad -\mathrm{Sin}x\,dx = du$

$e^x dx = dv \quad \downarrow \quad e^x = v$

$\mathrm{Sin}x = u \quad \downarrow \quad \mathrm{Cos}x\,dx = du$

$e^x dx = dv \quad \downarrow \quad e^x = v$

$$I = e^x\mathrm{Cos}x + e^x\mathrm{Sin}x - \underbrace{\int e^x \mathrm{Cos}x\,dx}_{I}$$

$$2I = e^x(\mathrm{Cos}x + \mathrm{Sin}x) \Rightarrow I = \frac{e^x}{2}(\mathrm{Cos}x + \mathrm{Sin}x) + C$$

$$\int_1^e x^3 \ln^2 x \, dx = \frac{x^4}{4}\ln^2 x\Big|_1^e - \frac{1}{2}\int_1^e x^3 \ln x\,dx = \frac{e^4}{4} - \frac{x^4}{4.2}\ln x\Big|_1^e + \frac{1}{8}\int_1^e x^3 dx$$

$\ln^2 x = u \quad \frac{2\ln x}{x}dx = du$

$x^3 dx = dv \quad \frac{x^4}{4} = v$

$\ln x = u \quad \downarrow \quad \frac{dx}{x} = du$

$x^3 dx = dv \quad \downarrow \quad \frac{x^4}{4} = v$

$$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{x^4}{32}\Big|_1^e = \frac{5e^4 - 1}{32}$$

(3) $\int \sin^m x \cdot \cos^n x \, dx$ integralleri

★ m çift, n tek ise: $u = \sin x$, $du = \cos x \, dx$

★ n çift, m tek ise: $u = \cos x$, $du = -\sin x \, dx$

dönüşümleri ve $\sin^2 x + \cos^2 x = 1$ özdeşliği kullanılarak integral çözülür.

★ Hem m, hem de n tek ise herhangi birine u denir

★ Hem m, hem de n çift ise: $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$ özdeşlikleri ile derece düşürülür

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(*) $\int \sin^3 x \cdot \cos^8 x \, dx = \int \underbrace{\sin^2 x}_{1-u^2} \cdot \underbrace{\cos^8 x}_{u^8} \cdot \underbrace{\sin x \, dx}_{-du} = \int u^8 (u^2 - 1) \, du$

$\cos x = u \quad -\sin x \, dx = du$

$= \frac{u^{11}}{11} - \frac{u^9}{9} + c$

$= \frac{(\cos x)^{11}}{11} - \frac{(\cos x)^9}{9} + c$

(*) $\int \cos^5(ax) \, dx = \int \underbrace{(\cos^2(ax))^2}_{(1-u^2)^2} \cdot \underbrace{\cos ax \, dx}_{\frac{du}{a}} = \frac{1}{a} \int (1 - 2u^2 + u^4) \, du$

$\sin ax = u$

$\downarrow$

$a \cos ax \, dx = du$

$= \frac{1}{a}\left(\sin x - \frac{2}{3}\sin^3 ax - \frac{1}{5}\sin^5 ax\right) + c$

(*) $\int \sin^5 x \cdot \cos^3 x \, dx = \int \underbrace{\sin^5 x}_{u^5} \cdot \underbrace{\cos^2 x}_{1-u^2} \cdot \underbrace{\cos x \, dx}_{du} = \int (u^5 - u^7) \, du = \frac{(\sin x)^6}{6} - \frac{(\sin x)^8}{8} + c$

$\sin x = u \quad \cos x \, dx = du$

(*) $\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \underbrace{\cos^2 2x}_{\frac{1+\cos 4x}{2}}) \, dx$

$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{4}\int \frac{1+\cos 4x}{2} \, dx$

$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + c$

④ $\int Sec^m x \cdot Tan^n x \, dx$ integralleri

a) $u = Secx \rightarrow du = Secx \, Tanx \, dx$
b) $u = Tanx \rightarrow du = Sec^2x \, dx$
c) Kısmi integrasyon

yollarından biri ve $Sec^2x = 1 + Tan^2x$ özdeşliği kullanılarak çözülür.

⊛ $\int Tan^2x \, dx = \int (Sec^2x - 1) \, dx = Tanx - x + c$

⊛ $$\int Sec^4x \, dx = \int \underbrace{Sec^2x}_{1+u^2} \cdot \underbrace{Sec^2x \, dx}_{du} = \int (1+u^2) du = u + \frac{u^3}{3} + c$$

$Tanx = u$
$\downarrow$
$Sec^2x \, dx = du$

$$= Tanx + \frac{(Tanx)^3}{3} + c$$

⊛ $$\int Sec^3x \cdot Tan^3x \, dx = \int \underbrace{Sec^2x}_{u^2} \cdot \underbrace{Tan^2x}_{u^2-1} \cdot \underbrace{Secx \cdot Tanx \cdot dx}_{du} = \int (u^4 - u^2) du$$

$Secx = u \rightarrow Secx \cdot Tanx \, dx = du$

$$= \frac{(Secx)^5}{5} - \frac{(Secx)^3}{3} + c$$

⊛ $$I = \int Sec^3x \, dx = \int Secx \cdot Sec^2x \, dx = Secx \, Tanx - \int Secx \cdot \underbrace{Tan^2x}_{Sec^2x - 1} \, dx$$

$Secx = u$ $\qquad$ $Sec^2x \, dx = dv$
$\downarrow$ $\qquad$ $\downarrow$
$Secx \, Tanx \, dx = du$ $\qquad$ $v = Tanx$

$$= Secx \, Tanx - \underbrace{\int Sec^3x \, dx}_{I} + \int Secx \, dx$$

$$\Downarrow$$

$$2I = Secx \, Tanx + \ln|Secx + Tanx|$$

$$I = \frac{1}{2}\left(Secx \, Tanx + \ln|Secx + Tanx|\right) + c$$

⑤ $\int \sin mx . \cos nx \, dx$ integralleri

$\sin mx . \cos nx$, $\sin mx . \sin nx$, $\cos mx . \cos nx$ çarpımlarını içeren integralleri çözmek için:

$$\sin mx . \cos nx = \frac{1}{2}\left(\sin(m-n)x + \sin(m+n)x\right)$$

$$\sin mx . \sin nx = \frac{1}{2}\left(\cos(m-n)x - \cos(m+n)x\right)$$

$$\cos mx . \cos nx = \frac{1}{2}\left(\cos(m-n)x + \cos(m+n)x\right)$$

dönüşümleri kullanılır.

(*) $$\int \sin 3x . \cos 5x \, dx = \frac{1}{2}\int \left(\sin(3x-5x) + \sin(3x+5x)\right) dx$$

$$= \frac{1}{2}\int \left(-\sin 2x + \sin 8x\right) dx = \frac{\cos 2x}{4} - \frac{\cos 8x}{16} + c$$

⑥ İndirgeme formülleri

(*) $\int \tan^n x \, dx$ integrali için bir indirgeme formülü bulup, bu formül yardımıyla $\int \tan^5 x \, dx$ integralini hesaplayınız.

$I_n = \int \tan^n x \, dx$ olsun.

$$I_n = \int (\tan x)^{n-2} . \tan^2 x \, dx = \int (\tan x)^{n-2} (\sec^2 x - 1) \, dx$$

$$= \int (\tan x)^{n-2} . \sec^2 x \, dx - \underbrace{\int (\tan x)^{n-2} dx}_{I_{n-2}}$$

$(\tan x = u \rightarrow \sec^2 x \, dx = du)$

$$= \int u^{n-2} . du - I_{n-2} = \frac{u^{n-1}}{n-1} - I_{n-2}$$

$$\boxed{I_n = \frac{(\tan x)^{n-1}}{n-1} - I_{n-2}}$$

$$I_n = \frac{(\tan x)^{n-1}}{n-1} - I_{n-2}$$

$$\Downarrow$$

$$I_5 = \int (\tan x)^5 dx = \frac{(\tan x)^4}{4} - I_3 = \frac{(\tan x)^4}{4} - \left(\frac{(\tan x)^2}{2} - I_1\right)$$

$$= \frac{(\tan x)^4}{4} - \frac{(\tan x)^2}{2} + \int \tan x$$

$$= \frac{(\tan x)^4}{4} - \frac{(\tan x)^2}{2} - \ln|\cos x| + c$$

⊛ $\int x^n . e^{ax} dx$ $(a \neq 0)$ integrali için bir indirgeme formülü bulup $\int x^3 . e^{ax} dx$ integralini bu formül yardımı ile hesaplayın.

$I_n = \int x^n e^{ax} dx$ olsun.

$x^n = u \rightarrow n.x^{n-1}. dx = du$

$e^{ax} dx = dv \rightarrow \frac{e^{ax}}{a} = v$

$$I_n = \frac{1}{a} . x^n . e^{ax} - \frac{n}{a} \int x^{n-1} . e^{ax} dx = \frac{1}{a} . x^n . e^{ax} - \frac{n}{a} I_{n-1}$$

$$\boxed{I_n = \frac{1}{a} . x^n e^{ax} - \frac{n}{a} I_{n-1}}$$

$$I_3 = \int x^3 e^{ax} dx = \frac{1}{a} x^3 e^{ax} - \frac{3}{a} I_2 = \frac{1}{a} x^3 . e^{ax} - \frac{3}{a}\left(\frac{1}{a} x^2 e^{ax} - \frac{2}{a} I_1\right)$$

$$= \frac{1}{a} x^3 e^{ax} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^2}\left(\frac{1}{a} x e^{ax} - \frac{1}{a} I_0\right)$$

$$= \frac{1}{a} x^3 e^{ax} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^3} x e^{ax} - \frac{6}{a^3} \int e^{ax} dx$$

$$= \frac{1}{a} x^3 e^{ax} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^3} x e^{ax} - \frac{6}{a^4} e^{ax} + c$$

7 Trigonometrik Değişken Dönüşümleri

$x = a\,\mathrm{Sin}\theta$ dönüşümü:

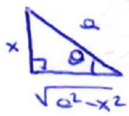
★ $\theta = \mathrm{ArcSin}\frac{x}{a}$ dönüşümüne denktir

★ $\sqrt{a^2-x^2}$ $(a>0)$ şeklinde terim içeren integrallerde kullanılır.

★ $\sqrt{a^2-x^2}$ ifadesi $-a \le x \le a$ için anlamlıdır; bu ise $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ için mümkündür.

★ $x = a\,\mathrm{Sin}\theta$ dönüşümü ile:

$$\boxed{\begin{array}{l} \sqrt{a^2-x^2} = \sqrt{a^2-a^2\mathrm{Sin}^2\theta} = a\,|\mathrm{Cos}\theta| = a\,\mathrm{Cos}\theta \\ x = a\,\mathrm{Sin}\theta \qquad dx = a\,\mathrm{Cos}\theta\, d\theta \end{array}}$$



$\Rightarrow$ $\mathrm{Cos}\theta = \frac{\sqrt{a^2-x^2}}{a}$

$\mathrm{Tan}\theta = \frac{x}{\sqrt{a^2-x^2}}$

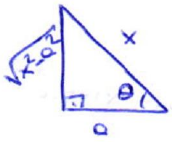
$x = a\,\mathrm{Tan}\theta$ dönüşümü

★ $\theta = \mathrm{ArcTan}\frac{x}{a}$ dönüşümüne denktir

★ $\sqrt{a^2+x^2}$ veya a^2+x^2 $(a>0)$ içeren integrallerde kullanılır. $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ için;

$$\boxed{\begin{array}{l} \sqrt{a^2+x^2} = \sqrt{a^2+a^2\mathrm{Tan}^2\theta} = a\,|\mathrm{Sec}\theta| = a\,\mathrm{Sec}\theta \\ x = a\,\mathrm{Tan}\theta \rightarrow dx = a\,\mathrm{Sec}^2\theta\, d\theta \end{array}}$$

$x = a\,\mathrm{Sec}\theta$ dönüşümü



★ $\theta = \mathrm{ArcSec}\frac{x}{a}$ dönüşümüne denktir

★ $\sqrt{x^2-a^2}$ içeren integrallerde kullanılır

★ $\mathrm{ArcSec}\theta$ nın tanım kümesi: $0 \le \theta < \frac{\pi}{2}$ ve $\frac{\pi}{2} < \theta \le \pi$ dir.

işlem kolaylığı için $0 \le \theta < \frac{\pi}{2}$ seçersek:

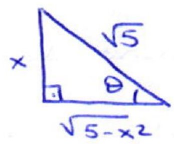
$$\boxed{\begin{array}{l} \sqrt{x^2-a^2} = \sqrt{a^2\mathrm{Sec}^2\theta - a^2} = a\,|\mathrm{Tan}\theta| = a\,\mathrm{Tan}\theta \\ x = a\,\mathrm{Sec}\theta \rightarrow dx = a\,\mathrm{Sec}\theta\,\mathrm{Tan}\theta\, d\theta \end{array}}$$

$$\circledast \int \frac{dx}{(5-x^2)^{3/2}} = \int \frac{\sqrt{5}\,\cos\theta\, d\theta}{5\sqrt{5}\,\cos^3\theta} = \frac{1}{5}\int \sec^2\theta\, d\theta = \frac{1}{5}\tan\theta + c$$

$x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta\, d\theta$

$\sqrt{5-x^2} = \sqrt{5(1-\sin^2\theta)} = \sqrt{5}\cos\theta$



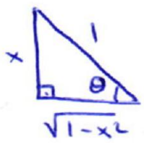
$$= \frac{1}{5}\,\frac{x}{\sqrt{5-x^2}} + c$$

$$\circledast \int \frac{x^2}{\sqrt{1-x^2}}\, dx = \int \frac{\sin^2\theta}{\cancel{\cos\theta}}\cancel{\cos\theta}\, d\theta = \int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}\, d\theta$$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$\sqrt{1-x^2} = \cos\theta$



$$= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + c$$

$$= \frac{\theta}{2} - \frac{\sin\theta . \cos\theta}{2} + c$$

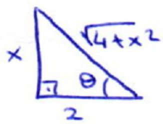
$$= \frac{\mathrm{ArcSin}\,x}{2} - \frac{1}{2} . x . \sqrt{1-x^2} + c$$

$$\circledast \int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$\sqrt{4+x^2} = 2\sec\theta$



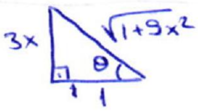
$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

$$\circledast \int \frac{dx}{(1+9x^2)^2} = \int \frac{1}{3} . \frac{\sec^2\theta}{\sec^4\theta}\, d\theta = \frac{1}{3}\int \cos^2\theta\, d\theta = \frac{1}{3}\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$3x = \tan\theta$

$3dx = \sec^2\theta\, d\theta$

$1+9x^2 = \sec^2\theta$



$$= \frac{1}{6}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{\theta}{6} + \frac{1}{6}\sin\theta\cos\theta + c$$

$$= \frac{\mathrm{ArcTan}\,x}{6} + \frac{1}{6} . \frac{3x}{\sqrt{1+9x^2}} . \frac{1}{\sqrt{1+9x^2}} + c$$

$$\circledast \int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta}{2\sec\theta . 2\tan\theta}\, d\theta = \int \frac{d\theta}{2} = \frac{1}{2}\theta + c = \frac{1}{2}\mathrm{ArcSec}\,\frac{x}{2} + c$$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-4} = \sqrt{4(\sec^2\theta - 1)} = 2\tan\theta$

⑧ Rasyonel fonksiyonların Basit Kesirlerle İntegrasyonu

Rasyonel fonksiyonların daha basit kesirlerin toplamı olarak yazma işlemine "Basit Kesirler Yöntemi" denir.

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3} \qquad A, B \rightarrow \text{Belirsiz Katsayılar}$$

* $\frac{P(x)}{Q(x)}$ rasyonel fonksiyonunu basit kesirlere ayırabilmek için: $d(P(x)) < d(Q(x))$ olmalıdır. Değilse polinom bölmesi ile bu hale getirilir.

Basit Kesirlere Ayırma Yöntemi:

① $x-r$, $Q(x)$ in lineer (1. dereceden) çarpanı olsun. $(x-r)^m$, $Q(x)$ i bölen $(x-r)$ nin en büyük kuvveti olsun. O zaman bu çarpan için m tane kısmi kesirin toplamı yazılır:

$$\boxed{(x-r)^m \text{ çarpanı için}: \frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}}$$

Bu işlem, $Q(x)$ in her lineer çarpanı için yapılır.

---

② x^2+px+q, $Q(x)$ in kuadratik (2. dereceden) çarpanı olsun. $(x^2+px+q)^n$, bu çarpanın $Q(x)$ i bölen en büyük kuvveti olsun.

$$\boxed{(x^2+px+q)^n \text{ için}: \frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}}$$ kısmi kesirleri yazılır.

---

③ Orjinal $\frac{P(x)}{Q(x)}$ kesri, tüm bu basit kesirlerin toplamına eşitlenir. Paydalar eşitlenip, paylar da polinom eşitliği yardımıyla katsayılar hesaplanır.

* Kapama (Heaviside) Yöntemi

$d(P(x)) < d(Q(x))$ ve $Q(x) = (x-r_1)\cdot(x-r_2)\cdots(x-r_n)$, $Q(x)$ in birbirinden farklı , kuvveti 1 olan lineer çarpanları ise:

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n}$$ olur ve

$$A_i = \lim_{x\to r_i} (x-r_i)\cdot\frac{P(x)}{(x-r_1)\cdots(x-r_n)}$$ dir. (Pratikte bunu, $\frac{P(x)}{Q(x)}$ in paydasındaki $x-r_i$ çarpanını çıkarıp kalan ifadede $x=r_i$ yazarak yaparız.)

(*) $\displaystyle\int \frac{5x-3}{x^2-2x-3}\,dx = ?$

$$\frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \qquad A = \left.\frac{5x-3}{x+1}\right|_{x=3} = 3 \qquad B = \left.\frac{5x-3}{x-3}\right|_{x=-1} = 2$$

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int\left(\frac{3}{x-3} + \frac{2}{x+1}\right)dx = 3\ln|x-1| + 2\ln|x+1| + C$$

(*) $\displaystyle I = \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = ?$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$A = \left.\frac{x^2+4x+1}{(x+1)(x+3)}\right|_{x=1} = \frac{6}{2.4} = \frac{3}{4}$$

$$B = \left.\frac{x^2+4x+1}{(x-1)(x+3)}\right|_{x=-1} = \frac{-2}{-2.2} = \frac{1}{2}$$

$$C = \left.\frac{x^2+4x+1}{(x-1)(x+1)}\right|_{x=-3} = \frac{9-12+1}{-4.-2} = -\frac{1}{4}$$

$$I = \int\left(\frac{3}{4}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{4}\cdot\frac{1}{x+3}\right)dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C$$

⊛ $\int \frac{6x+7}{(x+2)^2}\,dx = ?$

$\boxed{A=6}$

$2A+B=7 \Rightarrow \boxed{B=-5}$

$$\frac{6x+7}{(x+2)^2} = \underset{(x+2)}{\frac{A}{x+2}} + \frac{B}{(x+2)^2} = \frac{Ax+2A+B}{(x+2)^2}$$

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int\left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right)dx = 6\ln|x+2| + \frac{5}{(x+2)} + c$$

⊛ $\int \frac{x^3-4x^2-4x-3}{x^2-4x-5}\,dx = ?$

$$\frac{\overbrace{x^3-4x^2-4x-x}^{x(x^2-4x-5)}+x-3}{x^2-4x-5} = x + \frac{x-3}{x^2-4x-5}$$

$$\frac{x-3}{x^2-4x-5} = \frac{A}{x-5} + \frac{B}{x+1} \Rightarrow A = \frac{1}{3} \quad B = \frac{2}{3}$$

$$I = \int\left(x + \frac{1}{3}\cdot\frac{1}{x-3} + \frac{2}{3}\cdot\frac{1}{x+1}\right)dx = \frac{x^2}{2} + \frac{1}{3}\ln|x-3| + \frac{2}{3}\ln|x+1| + c$$

⊛ $I = \int \frac{4-2x}{(x^2+1)(x-1)}\,dx = ?$

$$\frac{4-2x}{(x^2+1)(x-1)} = \underset{(x-1)}{\frac{Ax+C}{x^2+1}} + \underset{(x^2+1)}{\frac{B}{x-1}} = \frac{Ax^2-Ax+Bx^2+B+Cx-C}{(x-1)(x^2+1)} \Rightarrow$$

$A+B=0$

$C-A=-2$

$B-C=4$

$B=1 \quad A=-1 \quad C=-3$

$$I = \int\left(-\frac{x+3}{x^2+1} + \frac{1}{x-1}\right)dx = \int -\frac{x}{x^2+1}\,dx - \int \frac{3}{x^2+1}\,dx + \int \frac{dx}{x-1}$$

$$= -\frac{1}{2}\ln|x^2+1| - 3\,\mathrm{Arctan}\,x + \ln|x-1| + c$$

9) $\tan\frac{x}{2}$ Dönüşümü

Sinx veya Cosx içeren kesirli fonksiyonların integralini alırken kullanılır.

$\tan\frac{x}{2} = u$ dönüşümü ile:

$x = 2\,\text{Arctan}\,u \Rightarrow dx = \frac{2}{1+u^2}\,du$ olur.

$$\sin x = 2.\sin\frac{x}{2}.\cos\frac{x}{2} = 2.\frac{u}{\sqrt{1+u^2}}.\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{1+u^2} - \frac{u^2}{1+u^2} = \frac{1-u^2}{1+u^2}$$

ÖZET

$u = \tan\frac{x}{2}$ ile:

$dx = \frac{2du}{1+u^2}$

$\sin x = \frac{2u}{1+u^2}$

$\cos x = \frac{1-u^2}{1+u^2}$

$$\circledast \int \frac{dx}{1+\cos x} = \int \frac{\frac{2\,du}{1+u^2}}{1+\frac{1-u^2}{1+u^2}} = \int \frac{\frac{2}{\cancel{1+u^2}}}{\frac{2}{\cancel{1+u^2}}}\,du = \int du = u + c = \tan\frac{x}{2} + c$$

$\tan\frac{x}{2} = u \qquad dx = \frac{2du}{1+u^2}$

$\cos x = \frac{1-u^2}{1+u^2}$

$$\circledast \int \frac{dx}{1-\sin x+\cos x} = \int \frac{\frac{2\,du}{1+u^2}}{\frac{2(1-u)}{1+u^2}} = \int \frac{du}{1-u} = -\ln|1-u| + c = -\ln\left|1-\tan\frac{x}{2}\right| + c$$

$\tan\frac{x}{2} = u \qquad dx = \frac{2du}{1+u^2}$

$$1 - \sin x + \cos x = 1 - \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2} = \frac{2(1-u)}{1+u^2}$$